An integral formula for differential forms and applications to Gaffney inequality

Abstract

A very old result, already known in the 19th century called the Helmholtz decomposition, states that every vector field f in $\mathbb{R}^3$ can be decomposed as

$$f = \operatorname{grad}\alpha + \operatorname{curl}\beta.$$

A far reaching generalization of this result, which takes into account certain boundary conditions, is the so called Hodge-Morrey decomposition. One of the crucial steps in the proof of this decomposition, which is due to Morrey , is the so called Gaffney inequality, which assumes the following form in the special case of vector fields in $\mathbb{R}^3$:

$$\|\omega\|^2_{W^{1,2}(\Omega)} \le C_\Omega \left(\|\operatorname{curl}\omega\|_{L^2(\Omega)} + \|\operatorname{div}\omega\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}\right)$$

for all vector fields ω satisfying certain boundary conditions on $\partial\Omega$. We recall that for vector fields ω in $\mathbb{R}^3$ we can identify $\operatorname{curl}\omega$ with the exterior derivative $d\omega$, respectively $\operatorname{div}\omega$ with the codifferential $\delta\omega$ (considering ω as a $1-$form). Our main result is a kind of generalization of the above inequality: given two $k-$forms α and β we derive an identity relating

$$\int_\Omega \left(\langle d\alpha; d\beta\rangle + \langle \delta\alpha; \delta\beta\rangle - \langle \nabla\alpha; \nabla\beta\rangle\right)$$

to an integral on the boundary of the domain and involving only the tangential and the normal components of α and β. We use this identity to deduce in a very simple way the classical Gaffney inequality and a generalization of it. This is joint work with B. Dacorogna.