Abstract

We present the optimal basis (OB) problem and the OB algorithm that we proposed in a published paper. The OB problem is formulated as follows. Given $m+1$ points $\{x_i\}_0^m$ in R^n which generate an m-dimensional linear manifold, construct for this manifold a maximally linearly independent basis that consists of vectors of the form $x_i - x_j$. This problem is present in, e.g., stable variants of the secant and interpolation methods, where it is required to approximate the Jacobian matrix f' of a nonlinear mapping f by using values of f computed at $m+1$ points. In this case, it is also desirable to have a combination of finite differences with maximal linear independence. As a natural measure of linear independence, we consider the Hadamard condition number which is minimized to find an optimal combination of m pairs $\{x_i, x_j\}$ that defines the optimal basis. This problem is not NP–hard, but can be reduced to the minimum spanning tree problem, which is solved by the greedy algorithm in $O(m^2)$ time. The complexity of this reduction is equivalent to one $m \times n$ matrix–matrix multiplication, and according to the Coppersmith–Winograd estimate, is below $O(n^{2.376})$ for $m = n$. We discuss possible applications of the OB algorithm for constructing simple non-diagonal prescaling procedures for iterative linear algebra solvers.