# BIVARIANT CYCLIC COHOMOLOGY AND CONNES' BILINEAR PAIRINGS IN NON-COMMUTATIVE MOTIVES

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Abstract. In this article we further the study of non-commutative motives, initiated in [3, 4, 25]. We prove that bivariant cyclic cohomology (and its variants) becomes representable in the category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ of non-commutative motives. Furthermore, Connes' bilinear pairings correspond to the composition operation in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$. As an application, we obtain a simple model, given in terms of infinite matrices, for the (de)suspension of these bivariant cohomology theories.

## 1. Introduction and statement of results

1.1. **Non-commutative motives.** A *differential graded (=dg) category*, over a commutative base ring $k$, is a category enriched over complexes of $k$-modules (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Dg categories enhance and solve many of the technical problems inherent to triangulated categories; see Keller's ICM adress [15]. In *non-commutative algebraic geometry* in the sense of Drinfeld, Kaledin, Kontsevich, and others [7, 8, 12, 17, 18, 19], dg categories are considered as dg-enhancements of bounded derived categories of (quasi-)coherent sheaves on a hypothetic non-commutative space.

All the classical invariants such as cyclic homology (and its variants), algebraic $K$-theory, and even topological cyclic homology, extend naturally from $k$-algebras to dg categories. In order to study all these invariants simultaneously the author introduced in [25] the notion of *localizing invariant*. This notion, makes use of the language of Grothendieck derivators (see §2.2), a formalism which allows us to state and prove precise universal properties. Let $L : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ be a morphism of derivators, from the derivator associated to the derived Morita model structure on dg categories (see §2.1), to a triangulated derivator. We say that $L$ is a localizing invariant if it preserves filtered homotopy and sends exact sequences of dg categories

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \quad \mapsto \quad L(\mathcal{A}) \longrightarrow L(\mathcal{B}) \longrightarrow L(\mathcal{C}) \longrightarrow L(\mathcal{A})[1]$$

to distinguished triangles in the base category $\mathbb{D}(e)$ of $\mathbb{D}$. Thanks to the work of Keller [16], Thomason-Trobaugh [30], and Blumberg-Mandell [2] (see also [29]) all the mentioned invariants[1] give rise to localizing invariants. In [25] the *universal*

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[1] In the case of algebraic $K$-theory we consider its non-connective version.

*localizing invariant* was constructed

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}} \,.$$

Given any triangulated derivator $\mathbb{D}$ we have an induced equivalence of categories

$$(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}, \mathbb{D}) \stackrel{\sim}{\longrightarrow} \underline{\mathsf{Hom}}_{\mathsf{loc}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}) \,, \tag{1.1}$$

where the left-hand side denotes the category of homotopy colimit preserving morphisms of derivators, and the right-hand side denotes the category of localizing invariants. Because of this universality property, which is a reminiscence of motives, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ is called the *localizing motivator*, and its base (triangulated) category $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ the category of *non-commutative motives*. We invite the reader to consult [1, 3, 4, 25, 28] for applications of this theory of non-commutative motives to the Farrell-Jones isomorphism conjectures, to Kontsevich's non-commutative mixed motives, to the construction of higher Chern characters, etc.

A fundamental problem in the theory of non-commutative motives is the computation of morphisms in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ and the description of its composition operation. In [3, 4, 25] an important step towards the solution of this problem was taken: let $\mathcal{A}$ be a *saturated* dg category in the sense of Kontsevich [17, 18], *i.e.* its complexes of morphisms are perfect and $\mathcal{A}$ is perfect as a bimodule over itself. Dg categories of perfect complexes associated to smooth and proper schemes and algebras of finite global cohomological dimension are classical examples. Then, for any $n \in \mathbb{Z}$ and dg category $\mathcal{B}$ we have a natural isomorphism in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$

$$\mathsf{Hom}\left(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B})[-n]\right) \simeq I\!\!K_n(\mathsf{rep}(\mathcal{A}, \mathcal{B})) \,, \tag{1.2}$$

where $I\!\!K$ denotes non-connective $K$-theory and $\mathsf{rep}(-,-)$ the internal Hom-functor in the homotopy category of dg categories (see §2.1). The composition operation is induced by the tensor product of bimodules. In particular, when $\mathcal{A}$ is the dg category $\underline{k}$ associated to the base ring $k$ (with one object and $k$ as the dg algebra of endomorphisms) we obtain

$$\mathsf{Hom}\left(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B})[-n]\right) \simeq I\!\!K_n(\mathcal{B}) \,. \tag{1.3}$$

At this point it is natural to ask the following motivational questions:

**Question A:** *Which (further) invariants of dg categories can be expressed in terms of morphism sets in the category of non-commutative motives?*

**Question B:** *How to explicitly describe the composition operation in these new cases?*

Intuitively, our answer is "Bivariant cyclic cohomology, with Connes' bilinear pairings playing the role of the composition operation".

1.2. **Bivariant cyclic cohomology.** Jones and Kassel, by drawing inspiration from Kasparov's $KK$-theory [13], introduced in [11] the bivariant cyclic cohomology theory of unital associative $k$-algebras. One of the fundamental properties of this bivariant theory is the fact that it simultaneously extends both negative cyclic homology as well as cyclic cohomology. Bivariant cyclic cohomology $HC^*(-,-)$ and its two variants, bivariant Hochschild cohomology $HH^*(-,-)$ [20, §5.5] and bivariant periodic cyclic cohomology $HP^*(-,-)$ [11, §8], extend naturally from $k$-algebras to dg categories. They associate to any pair of dg categories $(\mathcal{B}, \mathcal{C})$ a $\mathbb{Z}$-graded $k$-module which is contravariant on $\mathcal{B}$ and covariant on $\mathcal{C}$.

Our answer to the above Question A is the following.

**Theorem A.** *There exist triangulated functors*

$$\mathcal{T}^H, \mathcal{T}^C, \mathcal{T}^P : \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \longrightarrow \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \tag{1.4}$$

*such that for any $m \in \mathbb{Z}$ and dg categories $\mathcal{B}$ and $\mathcal{C}$, we have natural isomorphisms:*

$$\mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^H(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) \simeq HH^m(\mathcal{B}, \mathcal{C}) \tag{1.5}$$

$$\mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^C(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) \simeq HC^m(\mathcal{B}, \mathcal{C}) \tag{1.6}$$

$$\mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^P(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) \simeq HP^m(\mathcal{B}, \mathcal{C}), \tag{1.7}$$

*where (1.7) holds under the hypothesis that $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})$ is compact [22] in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$.*

Thanks to [4, Corollary 7.7] if $\mathcal{B}$ is a saturated dg category in the sense of Kontsevich, then $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})$ is compact object in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$. Recall from [20, §5.5.1][11] that we have the following isomorphisms:

$$HH^*(\underline{k}, \mathcal{C}) \simeq HH_{-*}(\mathcal{C}) \qquad HH^*(\mathcal{B}, \underline{k}) \simeq HH^*(\mathcal{B}) \tag{1.8}$$

$$HC^*(\underline{k}, \mathcal{C}) \simeq HC^-_{-*}(\mathcal{C}) \qquad HC^*(\mathcal{B}, \underline{k}) \simeq HC^*(\mathcal{B}) \tag{1.9}$$

$$HP^*(\underline{k}, \mathcal{C}) \simeq HP_{-*}(\mathcal{C}) \qquad HP^*(\mathcal{B}, \underline{k}) \simeq HP^*(\mathcal{B}). \tag{1.10}$$

Therefore, by replacing $\mathcal{B}$ or $\mathcal{C}$ by $\underline{k}$ we obtain two important instantiations of Theorem A: if we replace $\mathcal{C}$ by $\underline{k}$, the Hochschild $HH^*(-)$, cyclic $HC^*(-)$, and periodic cyclic $HP^*(-)$, cohomology theories become representable in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$; on the other hand if we replace $\mathcal{B}$ by $\underline{k}$, the above isomorphisms (1.8)-(1.10) combined with isomorphism (1.3) show us that the triangulated functors (1.4) allow us to "switch", inside the category of non-commutative motives, from algebraic $K$-theory to the Hochschild, negative cyclic, and periodic cyclic, homology.

**Connes' bilinear pairings.** In his foundational work on non-commutative geometry, in the early eighties, Connes [6] discovered bilinear pairings

$$\langle -, - \rangle : K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \longrightarrow k \qquad j \geq 0 \tag{1.11}$$

relating the Grothendieck group with the even part of cyclic cohomology. These bilinear pairings, which were the main motivation behind the development of a cyclic theory, consist roughly on the evaluation of a cyclic cochain at an idempotent representing a finitely generated projective module over $\mathcal{B}$.

Now, the above isomorphisms (1.2)-(1.3) and (1.5)-(1.7) show us that both algebraic $K$-theory as well as the different bivariant cohomology theories can be expressed in terms of morphism sets in the category of non-commutative motives. Therefore, the composition operation in the triangulated category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$, combined with these isomorphisms, furnish us bilinear pairings:

$$I\!K_n(\mathsf{rep}(\mathcal{A}, \mathcal{B})) \times HH^m(\mathcal{B}, \mathcal{C}) \longrightarrow HH^{m-n}(\mathcal{A}, \mathcal{C}) \tag{1.12}$$

$$I\!K_n(\mathsf{rep}(\mathcal{A}, \mathcal{B})) \times HC^m(\mathcal{B}, \mathcal{C}) \longrightarrow HC^{m-n}(\mathcal{A}, \mathcal{C}) \tag{1.13}$$

$$I\!K_n(\mathsf{rep}(\mathcal{A}, \mathcal{B})) \times HP^m(\mathcal{B}, \mathcal{C}) \longrightarrow HP^{m-n}(\mathcal{A}, \mathcal{C}). \tag{1.14}$$

Our answer to the above Question B is the following:

**Theorem B.** *The bilinear pairing (1.13), with $n = 0$, $\mathcal{A} = \mathcal{C} = \underline{k}$ and $m = 2j$, corresponds to Connes' original bilinear pairing (1.11).*

Theorem B supports the Grothendieckian belief that all classical constructions in (non-commutative) geometry should become conceptually clear in the correct category of (non-commutative) motives. The above pairings (1.12)-(1.14), which correspond to the composition operation in the category of non-commutative motives, can therefore be considered as an extension of Connes' foundational work.

## 2. Background on dg categories and derivators

Throughout this article $k$ denotes a commutative base ring with unit $\mathbf{1}$. Adjunctions are displayed vertically with the left (resp. right) adjoint on the left- (resp. right-) hand side.

2.1. **Dg categories.** Let $\mathcal{C}(k)$ be the category of (unbounded) complexes of $k$-modules. A *differential graded (=dg) category* is a category enriched over $\mathcal{C}(k)$ and a *dg functor* is a functor enriched over $\mathcal{C}(k)$; consult Keller's survey [15]. The category of dg categories will be denoted by $\mathsf{dgcat}$.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$ their *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is defined as follows: the set of objects is the cartesian product and given objects $(x, z)$ and $(y, w)$ in $\mathcal{A} \otimes \mathcal{B}$, we set $(\mathcal{A} \otimes \mathcal{B})((x, z), (y, w)) := \mathcal{A}(x, y) \otimes \mathcal{B}(z, w)$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a called a *derived Morita equivalence* if it induces an equivalence $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ between the associated derived categories. Thanks to [26, Theorem 5.3] the category $\mathsf{dgcat}$ carries a (cofibrantly generated) Quillen model structure [23] whose weak equivalences are the derived Morita equivalences. We denote by $\mathsf{Hmo}$ the homotopy category hence obtained.

The tensor product of dg categories can be derived into a bifunctor $- \otimes^{\mathbb{L}} -$ on $\mathsf{Hmo}$. Moreover, this bifunctor admits an internal Hom-functor $\mathsf{rep}(-, -)$. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ is the full dg subcategory of $\mathcal{A}$-$\mathcal{B}$-bimodules $X$ such that, for every object $x$ in $\mathcal{A}$, the right $\mathcal{B}$-module $X(-, x)$ is a compact object in the triangulated category $\mathcal{D}(\mathcal{B})$; see [4, §2.4].

2.2. **Grothendieck derivators.** Derivators allow us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using Quillen model categories; consult Grothendieck's original manuscript [9] or [5, §1] for a short account. Given a Quillen model category $\mathcal{M}$, we will denote by $\mathsf{HO}(\mathcal{M})$ its associated derivator. In order to simplify the exposition, a morphism of derivators and its value at the base category $e$ (which has one object and one morphism) will be denoted by the same symbol. It will be clear from the context which situation we are referring to.

## 3. Proof of Theorem A

We start by proving that the category of non-commutative motives satisfies an important compactness property.

**Proposition 3.1.** *The triangulated category* $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ *of non-commutative motives is well-generated in the sense of Neeman* [22, Remark 8.1.7].

*Proof.* Recall from [4, §7.1] that the triangulated category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ can be realized as the homotopy category of a stable Quillen model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$. This model category is obtained by taking left Bousfield localizations of presheaves of symmetric spectra on a small category $\mathsf{dgcat}_{\mathsf{f}}$; see the proof of [4, Theorem 7.5]. Since symmetric spectra [10] is a combinatorial model category and this class of model categories is stable under left Bousfield localizations and passage to presheaves, the model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ is combinatorial in the sense of Smith. Therefore, thanks to [24, Proposition 6.10] we conclude that $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ is well-generated in the sense of Neeman. $\surd$

Now, recall from [4, Examples 7.9 and 7.10] the construction of the Hochschild homology and mixed complex localizing invariants

$$HH : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathcal{C}(k)) \qquad C : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathcal{C}(\Lambda))\,.$$

Here, $\Lambda$ stands for the dg algebra $k[\epsilon]/\epsilon^2$, where the variable $\epsilon$ is of degree $-1$ and satisfies $d(\epsilon) = 0$. Thanks to the equivalence of categories (1.1), these morphisms of derivators factor (uniquely) through the universal localizing invariant $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}$. We obtain then triangulated functors

$$HH_{\mathsf{loc}} : \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \longrightarrow \mathcal{D}(k) \qquad C_{\mathsf{loc}} : \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \longrightarrow \mathcal{D}(\Lambda)\,, \tag{3.2}$$

and natural equivalences

$$HH \simeq HH_{\mathsf{loc}} \circ \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} \qquad C \simeq C_{\mathsf{loc}} \circ \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}\,. \tag{3.3}$$

**Proposition 3.4.** *The triangulated functors (3.2) admit right adjoints.*

*Proof.* The proof will consist on verifying the conditions of [22, Theorem 8.4.4]. First, observe that the triangulated categories $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$, $\mathcal{D}(k)$ and $\mathcal{D}(\Lambda)$ have small Hom-sets since they can be realized as homotopy categories of (stable) Quillen model categories. Thanks to Proposition 3.1 the category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ is well-generated in the sense of Neeman and so by [22, Theorem 1.17] it satisfies the representability theorem. Finally, since the left-hand side of the equivalence of categories (1.1) consists of homotopy colimit preserving morphisms of derivators, we conclude that the triangulated functors (3.2) respect arbitrary coproducts. This achieves the proof. $\surd$

Let us denote by

$$\begin{array}{ccc} \mathcal{D}(k) & & \mathcal{D}(\Lambda) \\ {\scriptstyle HH_{\mathsf{loc}}}\uparrow \ \downarrow{\scriptstyle R_H} & & {\scriptstyle C_{\mathsf{loc}}}\uparrow \ \downarrow{\scriptstyle R_C} \\ \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) & & \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \end{array} \tag{3.5}$$

the adjunctions of triangulated categories given by Proposition 3.4. The triangulated endofunctors $R_H \circ HH_{\mathsf{loc}}$ and $R_C \circ C_{\mathsf{loc}}$ of $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ will be denoted, respectively, by $\mathcal{T}^H$ and $\mathcal{T}^C$.

For any $m \in \mathbb{Z}$ and dg categories $\mathcal{B}$ and $\mathcal{C}$, we have natural isomorphisms:

$$\mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^H(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) \simeq \mathsf{Hom}_{\mathcal{D}(k)}(HH(\mathcal{B})[-m], HH(\mathcal{C})) \tag{3.6}$$

$$\simeq HH^m(\mathcal{B}, \mathcal{C})\,. \tag{3.7}$$

Isomorphism (3.6) follows from the left-hand side adjunction in (3.5) and from the left-hand side equivalence in (3.3). Isomorphism (3.7) follows from [20, Definition 5.5.5.1][2] and from the fact that homology in degree $-m$ of the complex of maps between $HH(\mathcal{B})$ and $HH(\mathcal{C})$ is naturally isomorphic to the set of maps in

[2] In *loc. cit.* the author used the symbol $C$, instead of $HH$, to denote the Hochschild complex.

$\mathcal{D}(k)$ between $HH(\mathcal{B})[-m]$ and $HH(\mathcal{C})$. This proves isomorphism (1.5) in Theorem A.

In the mixed complex case we have similar natural isomorphisms:

$$\begin{aligned}
&(3.8) && \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^C(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) && \simeq && \mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\mathcal{B})[-m], C(\mathcal{C}))\\
&(3.9) && && \simeq && \mathrm{Ext}^m_\Lambda(C(\mathcal{B}), C(\mathcal{C}))\\
&(3.10) && && = && HC^m(\mathcal{B}, \mathcal{C})\,.
\end{aligned}$$

Isomorphism (3.8) follow from the righ-hand side adjunction in (3.5) and from the right-hand side equivalence in (3.3). Isomorphism (3.9) is a standard fact in homological algebra. Equality (3.10) follows from the definition of bivariant cyclic cohomology; see [11, page 2]. This proves isomorphism (1.6) in Theorem A.

Let us now prove isomorphism (1.7) in Theorem A. Given any object $M$ in $\mathcal{D}(\Lambda)$, we have a functorial periodicity map $S$ in $\mathrm{Ext}^2_\Lambda(M, M) \simeq \mathsf{Hom}_{\mathcal{D}(\Lambda)}(M, M[2])$; see [11, §1]. This gives rise to an induced natural transformation of triangulated functors

$$S : \mathcal{T}^C = (R_C \circ C_{\mathsf{loc}}) \Rightarrow (R_C \circ (-[2]) \circ C_{\mathsf{loc}}) =: \mathcal{T}^C[2]\,.$$

Let $\mathcal{T}^P$ be the homotopy colimit [22, Definition 1.6.4] of the following diagram of natural transformations of triangulated functors

$$\mathcal{T}^C \stackrel{S}{\Rightarrow} \mathcal{T}^C[2] \stackrel{S}{\Rightarrow} \cdots \stackrel{S}{\Rightarrow} \mathcal{T}^C[2r] \stackrel{S}{\Rightarrow} \cdots\,.$$

For any $m \in \mathbb{Z}$ and dg categories $\mathcal{B}$ and $\mathcal{C}$, such that $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})$ is compact in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$, we have natural isomorphisms:

$$\begin{aligned}
& && \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^P(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right) && = && \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \varinjlim_r \mathcal{T}^C[2r](\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right)\\
&(3.11) && && \simeq && \varinjlim_r \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^C[2r](\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right)\\
&(3.12) && && \simeq && \varinjlim_r \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m], \mathcal{T}^C(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))[2r]\right)\\
& && && \simeq && \varinjlim_r \mathsf{Hom}\left(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})[-m-2r], \mathcal{T}^C(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{C}))\right)\\
&(3.13) && && \simeq && \varinjlim_r HC^{m+2r}(\mathcal{B}, \mathcal{C})\\
&(3.14) && && = && HP^m(\mathcal{B}, \mathcal{C})\,.
\end{aligned}$$

Isomorphism (3.11) follows from the compactness of $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{B})$ in the triangulated category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$. Isomorphism (3.12) follows from the natural equivalence of triangulated functors

$$\mathcal{T}^C[2r] := (R_C \circ (-[2r]) \circ C_{\mathsf{loc}}) \simeq (R_C \circ C_{\mathsf{loc}})[2r]\,.$$

Isomorphism (3.13) follows from the isomorphism (1.6) of Theorem A, which was already proven. Finally, equality (3.14) follows from the definition of bivariant periodic cyclic cohomology; see [11, Definition 8.1].

## 4. Proof of Theorem B

We start by describing the classical Chern character map in terms of the category of non-commutative motives. Recall from isomorphism (1.3) that, given any dg category $\mathcal{B}$, we have a natural isomorphism

$$K_0(\mathcal{B}) \simeq \mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B}))\,.$$

By combining isomorphism (1.9) with isomorphisms (3.9)-(3.10), we obtain also the following identification

$$\mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\mathcal{B})) \simeq HC_0^-(\mathcal{B})\,.$$

Therefore, since the triangulated functor

$$C_{\mathsf{loc}} : \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e) \longrightarrow \mathcal{D}(\Lambda)$$

sends $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})$ to $C(\underline{k})$, we obtain an induced map

$$K_0(\mathcal{B}) \simeq \mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B})) \longrightarrow \mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\mathcal{B})) \simeq HC_0^-(\mathcal{B})\,. \tag{4.1}$$

**Proposition 4.2.** *Given any dg category $\mathcal{B}$, the above map (4.1) is the Chern character map $ch^-(\mathcal{B})$ of $\mathcal{B}$; see [20, §8.3].*

*Proof.* Thanks to isomorphism (1.3) the Grothendieck group functor can be expressed as the following composition

$$K_0 : \mathsf{dgcat} \longrightarrow \mathsf{Hmo} \stackrel{\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}}{\longrightarrow} \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e) \stackrel{\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}),-)}{\longrightarrow} \mathsf{Ab}\,, \tag{4.3}$$

where $\mathsf{Ab}$ denotes the category of abelian groups. Similarly, isomorphism (1.8) combined with isomorphisms (3.9)-(3.10), allow us to express the degree zero negative cyclic homology group functor as the following composition

$$HC_0^- : \mathsf{dgcat} \longrightarrow \mathsf{Hmo} \stackrel{C}{\longrightarrow} \mathcal{D}(\Lambda) \stackrel{\mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}),-)}{\longrightarrow} \mathsf{Ab}\,. \tag{4.4}$$

Since we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Hmo} & \stackrel{C}{\longrightarrow} & \mathcal{D}(\Lambda) \\ {\scriptstyle \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}}\downarrow & \nearrow_{C_{\mathsf{loc}}} & \\ \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e) & & \end{array},$$

the triangulated functor $C_{\mathsf{loc}}$ gives rise to a natural transformation $K_0 \Rightarrow HC_0^-$ between the above functors (4.3)-(4.4), whose value at an arbitrary dg category $\mathcal{B}$ is the above map (4.1).

Now, recall from [28, Theorem 1.3] that there is an isomorphism

$$\mathrm{Nat}(K_0, HC_0^-) \stackrel{\sim}{\longrightarrow} HC_0^-(\underline{k}) \simeq k \qquad ch^- \mapsto \mathbf{1} \tag{4.5}$$

between natural transformations and the base ring $k$. This isomorphism is given by the evaluation of a natural transformation at the class $[k]$ of $k$ (as a module over itself) in the Grothendieck group $K_0(\underline{k}) = K_0(k)$. Moreover, under this isomorphism the Chern character map $ch^-$ corresponds to the unit $\mathbf{1}$ of the base ring $k$. The class $[k]$ of $k$ in $K_0(\underline{k})$ is given by the identity map in $\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}))$ and the identity map in $\mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k}))$ corresponds, under the natural isomorphisms

$$\mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k})) \simeq \mathrm{Ext}^0_\Lambda(C(\underline{k}), C(\underline{k})) = HC^0(\underline{k}, \underline{k}) \simeq HC_0^-(\underline{k}) \simeq k\,,$$

to the identity $\mathbf{1}$ of $k$; see [11, Theorem 2.3]. Therefore, the above isomorphism (4.5) allow us to conclude that given any dg category $\mathcal{B}$, the map (4.1) is in fact the Chern character map $ch^-(\mathcal{B})$ of $\mathcal{B}$. This achieves the proof. $\surd$

Now, recall from [20, §8.3.10] that Connes' bilinear pairings (1.11) can be expressed as the following compositions

$$(4.6) \qquad \langle -,-\rangle : K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \stackrel{ch_{2j}\times \mathrm{id}}{\longrightarrow} HC_{2j}(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \stackrel{\mathrm{ev}}{\longrightarrow} k \qquad j \geq 0\,,$$

where $ch_{2j}$ is the Chern character map (see [20, §8.3]) and ev is induced by the evaluation of cyclic cochains on cyclic chains. Thanks to the adjunction

$$\begin{array}{c} \mathcal{D}(\Lambda) \\ C_{\mathsf{loc}} \uparrow \quad \downarrow R_C \\ \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e) \end{array}$$

we obtain a commutative square[3]

$$\begin{array}{ccc} \mathsf{Hom}(\mathcal{U}(\underline{k}),\mathcal{U}(\mathcal{B})) \times \mathsf{Hom}(\mathcal{U}(\mathcal{B}),\mathcal{T}^C(\mathcal{U}(\underline{k}))[2j]) & \stackrel{\mathrm{comp}}{\longrightarrow} & \mathsf{Hom}(\mathcal{U}(\underline{k}),\mathcal{T}^C(\mathcal{U}(\underline{k}))[2j]) \\ \downarrow {\scriptstyle C_{\mathsf{loc}}\times\varphi} & & \sim \downarrow {\scriptstyle \varphi} \\ \mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}),C(\mathcal{B})) \times \mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\mathcal{B}),C(\underline{k})[2j]) & \underset{\mathrm{comp}}{\longrightarrow} & \mathsf{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}),C(\underline{k})[2j])\,, \end{array}$$

where $\varphi$ is the natural isomorphism given by the adjunction and the horizontal maps are the composition operations in $\mathcal{D}(\Lambda)$ and $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$. Proposition 4.2, combined with isomorphisms (1.6), (1.9) and (3.9)-(3.10), show us that the above commutative square correspond to the following diagram

$$(4.7) \qquad \begin{array}{ccc} K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \longrightarrow & HC^{2j}(\underline{k}) \simeq k \\ \downarrow {\scriptstyle ch^-(\mathcal{B})\times \mathrm{id}} & & \| \\ HC_0^-(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \underset{\mathrm{comp}}{\longrightarrow} & HC^{2j}(\underline{k}) \simeq k\,, \end{array}$$

where the upper horizontal map is the pairing (1.13) (with $n=0$, $\mathcal{A}=\mathcal{C}=\underline{k}$ and $m=2j$). Now, recall from [20, §5.1.8] that there exist natural maps

$$U_j : HC_0^-(\mathcal{B}) \longrightarrow HC_{2j}(\mathcal{B}) \qquad j \geq 0$$

such that $U_j \circ ch^-(\mathcal{B}) = ch_{2j}(\mathcal{B})$. Thanks to the description of the composition operation in $\mathcal{D}(\Lambda)$ given in [11, Theorem 5.1] we have the following diagram

$$(4.8) \qquad \begin{array}{ccc} HC_0^-(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \stackrel{\mathrm{comp}}{\longrightarrow} & k \\ \downarrow {\scriptstyle U_j\times \mathrm{id}} & & \| \\ HC_{2j}(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \underset{\mathrm{ev}}{\longrightarrow} & k\,. \end{array}$$

Finally, by combining diagram (4.7) with diagram (4.8) we conclude that the pairing (1.13) (with $n=0$, $\mathcal{A}=\mathcal{C}=\underline{k}$ and $m=2j$) identifies with the above composition (4.6), and so with Connes' original bilinear pairing (1.11). This achieves the proof.

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[3] In order to reduce the size of the diagram we have denoted, in the upper row, the universal localizing invariant $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}$ by $\mathcal{U}$.

## 5. An application: (de)suspension of bivariant cohomology theories

In [27] the author constructed a simple model for the suspension in the triangulated category of non-commutative motives. Consider the $k$-algebra $\Gamma$ of $\mathbb{N}\times\mathbb{N}$-matrices $A$ which satisfy the following two conditions: the set $\{A_{i,j}\,|\,i,j\in\mathbb{N}\}$ is finite and there exists a natural number $n_A$ such that each row and each column has at most $n_A$ non-zero entries. Let $\Sigma$ be the quotient of $\Gamma$ by the two-sided ideal consisting of those matrices with finitely many non-zero entries; see [27, §3]. Alternatively, consider the (left) localization of $\Gamma$ with respect to the matrices $\overline{I_n}, n\geq 0$, with entries $(\overline{I_n})_{i,j}=\mathbf{1}$ for $i=j>n$ and 0 otherwise. Then, for any dg category $\mathcal{A}$ we have a canonical isomorphism

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\Sigma(\mathcal{A}))\xrightarrow{\sim}\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})[1] \tag{5.1}$$

in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$, where $\Sigma(\mathcal{A})=\mathcal{A}\otimes\Sigma$. Note that (5.1) shows us that if $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B})$ is a compact object in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ then so it is $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\Sigma(\mathcal{B}))$. Now, by combining isomorphism (5.1) with Theorem A we obtain the following result.

**Theorem 5.2.** *For any dg categories $\mathcal{B}$ and $\mathcal{C}$, we have natural isomorphisms*

$$HH^{*+1}(\Sigma(\mathcal{B}),\mathcal{C})\simeq HH^*(\mathcal{B},\mathcal{C}) \qquad HH^{*-1}(\mathcal{B},\Sigma(\mathcal{C}))\simeq HH^*(\mathcal{B},\mathcal{C}) \tag{5.3}$$

$$HC^{*+1}(\Sigma(\mathcal{B}),\mathcal{C})\simeq HC^*(\mathcal{B},\mathcal{C}) \qquad HC^{*-1}(\mathcal{B},\Sigma(\mathcal{C}))\simeq HC^*(\mathcal{B},\mathcal{C}) \tag{5.4}$$

$$HP^{*+1}(\Sigma(\mathcal{B}),\mathcal{C})\simeq HP^*(\mathcal{B},\mathcal{C}) \qquad HP^{*-1}(\mathcal{B},\Sigma(\mathcal{C}))\simeq HP^*(\mathcal{B},\mathcal{C})\,, \tag{5.5}$$

*where the left-hand side of (5.5) holds under the hypothesis that $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{B})$ is compact (for instance if $\mathcal{B}$ is a saturated dg category in the sense of Kontsevich).*

Theorem 5.2 extends Kassel's previous work [14, §III Theorem 3.1] on bivariant cyclic cohomology on ordinary algebras defined over a field to dg categories defined over a general commutative base ring. Hence, it can now be applied to schemes. Given a (quasi-compact and quasi-separated) $k$-scheme $X$, it is well-known that the category of perfect complexes of $\mathcal{O}_X$-modules admits a dg-enhancement $\mathsf{perf}_{\mathsf{dg}}(X)$; see for instance [21] or [4, Example 4.5]. Moreover, whenever $X$ is smooth and proper the dg category $\mathsf{perf}_{\mathsf{dg}}(X)$ is saturated in the sense of Kontsevich. Given a pair $(X,Y)$ of $k$-schemes, the bivariant Hochschild, cyclic, and periodic, homology of $(X,Y)$ can be obtained from the pair of dg categories $(\mathsf{perf}_{\mathsf{dg}}(X),\mathsf{perf}_{\mathsf{dg}}(Y))$ by applying the corresponding bivariant theory. Therefore, when $\mathcal{B}=\mathsf{perf}_{\mathsf{dg}}(X)$ and $\mathcal{C}=\mathsf{perf}_{\mathsf{dg}}(Y)$, the above isomorphisms (5.3)-(5.5) reduce to the corresponding isomorphisms associated to the schemes $X$ and $Y$.

Note also that the isomorphisms (1.8)-(1.10) lead to important specializations of Theorem 5.2 when we replace $\mathcal{B}$ or $\mathcal{C}$ by $\underline{k}$. When $\mathcal{B}=\underline{k}$, $\Sigma(\mathcal{C})$ corresponds to the suspension of $\mathcal{C}$ in the different homology theories. When $\mathcal{C}=\underline{k}$, $\Sigma(\mathcal{B})$ corresponds to the desuspension.

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