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Relations between the Chow motive and the noncommutative motive of a smooth projective variety

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ABSTRACT

In this note we relate the notions of Lefschetz type, decomposability, and isomorphism for Chow motives with the notions of trivial type, decomposability, and isomorphism for noncommutative motives. Some examples, counter-examples, and applications are also described.

1. Introduction and statement of results

The goal of this note is not to introduce new concepts but rather to gather the work of different people in order to establish some precise relations between the recent theory of noncommutative motives and the classical theory of Chow motives. The key technical ingredient used is the Grothendieck–Riemann–Roch theorem. In what follows, k denotes a base field and R a commutative ring of coefficients.

Chow motives. In the early sixties Grothendieck envisioned the existence of a "universal" cohomology theory of schemes. Among several conjectures and developments, a contravariant $\otimes$-functor $M(-)_R$: $\mathrm{SmProj}(k)^{\mathrm{op}} \to \mathsf{Chow}(k)_R$ from smooth projective k-schemes to *Chow motives* (with R coefficients) was constructed. Intuitively speaking, $\mathsf{Chow}(k)_R$ encodes all the geometric/arithmetic information about smooth

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projective k-schemes and acts as a gateway between algebraic geometry and the assortment of the numerous Weil cohomology theories.

Noncommutative motives. A *dg category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; see Section 2.1. Every (dg) k-algebra A gives naturally rise to a dg category $\underline{A}$ with a single object. Another source of examples is provided by schemes since the bounded derived category $\mathcal{D}^b(X)$ of every smooth projective k-scheme X admits a unique dg enhancement $\mathcal{D}^b_{\mathsf{dg}}(X)$; see Lunts and Orlov [18]. In what follows, we will write $\mathrm{dgcat}(k)$ for the category of (small) dg categories.

All the classical invariants such as algebraic K-theory, cyclic homology, and topological Hochschild homology extend naturally from k-algebras to dg categories. In order to study all these invariants simultaneously the notion of additive invariant, which we now recall, was introduced in [25]. Let I be the dg category with objects $\{1, 2\}$ and complexes of morphisms $I(1, 1) = I(2, 2) = I(1, 2) = k$ and $I(2, 1) = 0$. Given a dg category $\mathcal{A}$, let us write $T(\mathcal{A})$ for the tensor product $\mathcal{A} \otimes I$. Intuitively speaking, $T(\mathcal{A})$ "dg categorifies" the notion of upper triangular matrix. Note that we have two inclusion dg functors $i_1, i_2 : \mathcal{A} \to T(\mathcal{A})$. Given an R-linear additive category D, a functor $E : \mathrm{dgcat}(k) \to \mathsf{D}$ is called an *additive invariant* if it satisfies the following two conditions:

(i) It sends Morita equivalences (see Section 2.1) to isomorphisms;
(ii) The dg functors ι_1, ι_2 induce an isomorphism[2] $E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A}))$.

Fix a commutative ring R. In [25], an R-linear additive category $\mathsf{Hmo}_0(k)_R$ and an additive invariant $U(-)_R : \mathrm{dgcat}(k) \to \mathsf{Hmo}_0(k)_R$ were constructed. This functor is characterized by the following universal property: given any R-linear additive category D, there is an induced equivalence of categories

$$U(-)^*_R : \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0(k)_R, \mathsf{D}) \xrightarrow{\sim} \mathrm{Inv}_{\mathsf{add}}(\mathrm{dgcat}(k), \mathsf{D})\,, \tag{1.1}$$

where the left-hand side denotes the category of R-linear additive functors and the right-hand side the category of additive invariants. Because of this universal property, $\mathsf{Hmo}_0(k)_R$ is called the category of *noncommutative motives*.

Statement of results. Let $\mathbf{L} \in \mathsf{Chow}(k)_R$ be the Lefschetz motive and $\mathbf{1} := U(\underline{k})_R$ the $\otimes$-unit of $\mathsf{Hmo}_0(k)_R$. Following Gorchinskiy and Orlov [9], a Chow motive is called of *Lefschetz type* if it is isomorphic to $\mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ for some non-negative integers $l_1, \ldots, l_m$. In the same vein, a noncommutative motive is called of *trivial type* if it is isomorphic to $\oplus_{i=1}^m \mathbf{1}$ for some integer m. The following implication was established by Gorchinskiy and Orlov in [9, §4] (assuming that $\mathbb{Z} \subseteq R$):

$$M(X)_R \text{ Lefschetz type} \Rightarrow U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \text{ trivial type}\,. \tag{1.2}$$

In the particular case where $R = \mathbb{Q}$, (1.2) becomes an equivalence (see [19, §1]):

$$M(X)_{\mathbb{Q}} \text{ Lefschetz type} \Leftrightarrow U(\mathcal{D}^b_{\mathsf{dg}}(X))_{\mathbb{Q}} \text{ trivial type}\,. \tag{1.3}$$

The following result establishes a partial converse of the above implication (1.2):

Theorem 1.4. *Let X be an irreducible smooth projective k-scheme of dimension d. Assume that $\mathbb{Z} \subseteq R$ and that every finitely generated projective $R[1/(2d)!]$-module is free (e.g. R a principal ideal domain). Assume*

[2] Condition (ii) can be equivalently formulated in terms of semi-orthogonal decompositions in the sense of Bondal and Orlov [4]; see [25, Thm. 6.3(4)].

also that $U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \simeq \oplus_{i=1}^m \mathbf{1}$ *for some integer* m*. Under these assumptions, there is a choice of integers (up to permutation)* $l_1, \dots, l_m \in \{0, \dots, d\}$ *giving rise to an isomorphism*

$$M(X)_{R[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} . \tag{1.5}$$

Intuitively speaking, Theorem 1.4 shows that the converse of the above implication (1.2) holds as soon as one inverts the integer $(2d)!$ (or equivalently its prime factors). By combining this result with (1.2), one obtains a refinement of (1.3):

Corollary 1.6. *Given* X *and* R *as in Theorem 1.4, we have the equivalence*

$$M(X)_{R[1/(2d)!]} \text{ Lefschetz type} \Leftrightarrow U(\mathcal{D}^b_{\mathsf{dg}}(X))_{R[1/(2d)!]} \text{ trivial type} .$$

In the particular case where X is a curve C or a surface S and $R = \mathbb{Z}$, we have

$$M(C)_{\mathbb{Z}[1/2]} \text{ Lefschetz type} \Leftrightarrow U(\mathcal{D}^b_{\mathsf{dg}}(C))_{\mathbb{Z}[1/2]} \text{ trivial type} .$$
$$M(S)_{\mathbb{Z}[1/6]} \text{ Lefschetz type} \Leftrightarrow U(\mathcal{D}^b_{\mathsf{dg}}(S))_{\mathbb{Z}[1/6]} \text{ trivial type} .$$

As the following proposition shows, the (strict) converse of implication (1.2) is false!

Proposition 1.7. *Let* q *be a non-singular quadratic form and* Q_q *the associated smooth projective quadric. Assume that* q *is even dimensional, anisotropic, and has trivial discriminant and trivial Clifford invariant (see Lam [17, §V.2]).*

(i) *The noncommutative motive* $U(\mathcal{D}^b_{\mathsf{dg}}(Q_q))_{\mathbb{Z}}$ *is of trivial type.*
(ii) *The Chow motive* $M(Q_q)_{\mathbb{Z}}$ *is not of Lefschetz type.*[3]

Example 1.8. As explained by Lam in [17, §V Cor. 3.4 and page 138], a non-singular quadratic form q is even dimensional, anisotropic, and has trivial discriminant and trivial Clifford invariant if and only if it belongs to the third power of the fundamental ideal $I(k)$ of the Witt ring $W(k)$.

As an application of Theorem 1.4, we obtain the following sharpening of the main result of [19] (which was obtained only with rational coefficients).

Corollary 1.9. *Let* X *be an irreducible smooth projective* k*-scheme of dimension* d*. Assume that* $\mathcal{D}^b(X)$ *admits a full exceptional collection* $(\mathcal{E}_1, \dots, \mathcal{E}_m)$ *of length* m*. Under these assumptions, there is a choice of integers (up to permutation)* $l_1, \dots, l_m \in \{0, \dots, d\}$ *giving rise to an isomorphism*

$$M(X)_{\mathbb{Z}[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} . \tag{1.10}$$

Thanks to the work of Beilinson, Kapranov, Kawamata, Kuznetsov, Orlov, and others (see [2,10,13,15,21]), Corollary 1.9 applies to projective spaces and rational surfaces (in the case where k is algebraically closed), and to smooth quadric hypersurfaces, Grassmannians, flag varieties, Fano threefolds with vanishing odd cohomology, and toric varieties (in the case where $k = \mathbb{C}$). Conjecturally, it applies also to all the homogeneous spaces of the form G/P, with P a parabolic subgroup of a semisimple algebraic group G; see Kuznetsov and Polishchuk [16].

The following result concerns decomposability:

[3] The explicit computation of $M(Q_q)_{\mathbb{Z}}$ was achieved by Rost [23]; see also Elman, Karpenko and Merkurjev [6, §XVII].

Theorem 1.11. *Let X be an irreducible smooth projective k-scheme of dimension d. Under the assumption $\mathbb{Z} \subseteq R$, the following implication holds:*

$$M(X)_{R[1/(2d)!]} \text{ decomposable} \Rightarrow U(\mathcal{D}^b_{\mathsf{dg}}(X))_{R[1/(2d)!]} \text{ decomposable}. \tag{1.12}$$

As the following proposition shows, if one does not invert the dimension of X, the converse of implication (1.12) is false!

Proposition 1.13. *Let A be a central simple k-algebra of degree $\sqrt{\dim(A)} = d$ and $X = \mathrm{SB}(A)$ the associated Severi–Brauer variety.*

(i) *For every commutative ring R one has the following motivic decomposition*

$$U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \simeq \mathbf{1} \oplus U(\underline{A})_R \oplus U(\underline{A})^{\otimes 2}_R \oplus \cdots \oplus U(\underline{A})^{\otimes d-1}_R . \tag{1.14}$$

In particular, the noncommutative motive $U(\mathcal{D}^b_{\mathsf{dg}}(X))_R$ is decomposable.

(ii) *(Karpenko) When A is a division algebra, the Chow motive $M(X)_{\mathbb{Z}}$ is indecomposable. Moreover, when d is a prime power p^s, the Chow motive $M(X)_{\mathbb{Z}/p\mathbb{Z}}$ is also indecomposable.*

Remark 1.15. Item (ii) holds also for $M(X)_{\mathbb{Z}_p}$; see De Clercq [5, Rmq. 2.3].

Roughly speaking, Proposition 1.13 shows that the decomposition (1.14) is “truly noncommutative”. Our final result is the following:

Theorem 1.16. *Let $\{X_i\}_{1\leq i\leq n}$ (resp. $\{Y_j\}_{1\leq j\leq m}$) be irreducible smooth projective k-schemes of dimension d_{X_i} (resp. d_{Y_j}), $d := \max\{d_{X_i}, d_{Y_j} \,|\, i, j\}$, and $\{l_i\}_{1\leq i\leq n}$ (resp. $\{l_j\}_{1\leq j\leq m}$) arbitrary integers. Assume that $\mathbb{Z} \subseteq R$ and $1/(2d)! \in R$. Under these assumptions, we have the following implication*

$$\oplus_i M(X_i)_R \otimes \mathbf{L}^{\otimes l_i} \simeq \oplus_j M(Y_j)_R \otimes \mathbf{L}^{\otimes l_j} \Rightarrow \oplus_i U(\mathcal{D}^b_{\mathsf{dg}}(X_i))_R \simeq \oplus_j U(\mathcal{D}^b_{\mathsf{dg}}(Y_j))_R .$$

As the following example shows, if one does not invert the maximum of the dimensions, the converse of the implication of Theorem 1.16 is false!

Example 1.17. The Chow motives $M(X)_{\mathbb{Z}}$ and $M(\widehat{X})_{\mathbb{Z}}$ of an abelian variety X and of its dual $\widehat{X}$ are in general not isomorphic. However, thanks to the work of Mukai [20], we have $U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \simeq U(\mathcal{D}^b_{\mathsf{dg}}(\widehat{X}))_R$ for every commutative ring R.

Finally, by combining Theorem 1.16 with (1.1), we obtain the application:

Corollary 1.18. *Let X (resp. Y) be an irreducible smooth projective k-scheme of dimension d_X (resp. d_Y), and $d := \max\{d_X, d_Y\}$. Assume that $\mathbb{Z} \subseteq R$ and $1/(2d)! \in R$. Under these assumptions, $M(X)_R \simeq M(Y)_R \Rightarrow E(X) \simeq E(Y)$ for every additive invariant E with values in an R-linear additive category.*

2. Preliminaries

2.1. Dg categories

A *differential graded (= dg) category* $\mathcal{A}$ is a category enriched over complexes of k-modules (morphisms sets $\mathcal{A}(x, y)$ are complexes) in such a way that composition fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$; consult Keller ICM survey [14]. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence of (triangulated) categories $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ between the associated derived categories; see [14, §4.6]. Finally, the *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories is defined as follows: the set of objects is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and the complexes of morphisms are given by $(\mathcal{A} \otimes \mathcal{B})((x,z),(y,w)) = \mathcal{A}(x,y) \otimes \mathcal{B}(z,w)$.

2.2. K_0-motives

Recall from Gillet and Soulé [7, §5.2], [8, §5.2] the construction of the category $\mathsf{KM}(k)_R$ of K_0-*motives*. The objects are the smooth projective k-schemes, the morphisms are given by $\mathrm{Hom}_{\mathsf{KM}(k)_R}(X,Y) := K_0(X \times Y)_R$, and the symmetric monoidal structure is induced by the product of k-schemes. Furthermore, $\mathsf{KM}(k)_R$ comes equipped with a canonical (contravariant) $\otimes$-functor

$$M_0(-)_R : \mathrm{SmProj}(k)^{\mathrm{op}} \longrightarrow \mathsf{KM}(k)_R \qquad X \mapsto X$$

that sends a morphism $f : X \to Y$ in $\mathrm{SmProj}(k)$ to the class $[\mathcal{O}_{\Gamma_f^t}] \in K_0(Y \times X)_R$ of the transpose Γ_f^t of the graph $\Gamma_f := \{(x, f(x)) \,|\, x \in X\} \subset X \times Y$ of f.

Notation 2.1. Given irreducible smooth projective k-schemes $X_1, \ldots, X_n$ of dimension $d_1, \ldots, d_n$, let us denote by $(X_1, \ldots, X_n)_R$ the full subcategory of $\mathsf{KM}(k)_R$ consisting of the objects $\{M_0(X_i)_R \,|\, 1 \le i \le n\}$. Its closure (inside $\mathsf{KM}(k)_R$) under finite direct sums will be denoted by $(X_1, \ldots, X_n)_R^{\oplus}$.

Remark 2.2. As explained in [19, §4.4], there exists an R-linear additive *fully faithful* $\otimes$-functor θ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{SmProj}(k)^{\mathrm{op}} & \xrightarrow{\mathcal{D}^b_{\mathrm{dg}}(-)} & \mathrm{dgcat}(k) \\ {\scriptstyle M_0(-)_R}\downarrow & & \downarrow{\scriptstyle U(-)_R} \\ \mathsf{KM}(k)_R & \xrightarrow[\theta]{} & \mathsf{Hmo}_0(k)_R\,. \end{array} \tag{2.3}$$

2.3. Orbit categories

Let $\mathcal{C}$ be an additive symmetric monoidal category and $\mathcal{O}$ a $\otimes$-invertible object. Recall from [24, §7] that the *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a,b) := \oplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b\otimes\mathcal{O}^{\otimes j})$. Given objects a, b, c and morphisms $\underline{f} = \{f_j\}_{j\in\mathbb{Z}} \in \mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a,b)$ and $\underline{g} = \{g_j\}_{j\in\mathbb{Z}} \in \mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(b,c)$, the lth-component of the composition $\underline{g} \circ \underline{f}$ is given by the finite sum $\sum_j ((g_{l-j} \otimes \mathcal{O}^{\otimes j}) \circ f_j)$. We obtain in this way an additive category $\mathcal{C}/_{-\otimes\mathcal{O}}$ and a canonical additive projection functor $\pi : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$ which comes equipped with a natural 2-isomorphism $\pi \circ (- \otimes \mathcal{O}) \stackrel{\sim}{\Rightarrow} \pi$.

3. Proof of Theorem 1.4

By assumption, $\mathbb{Z} \subsetneq R$. Let us write $R_{\mathbb{Q}}$ for the localization of R at $\mathbb{Z}\backslash\{0\}$. Recall that the Lefschetz motive $\mathbf{L}$ is a $\otimes$-invertible object. Following Section 2.3, we can then consider the orbit category $\mathsf{Chow}(k)_{R_{\mathbb{Q}}}/_{-\otimes\mathbf{L}}$.

Proposition 3.1. *There exists an additive functor* Ψ *making the diagram commute:*

$$\begin{array}{ccc} \mathrm{SmProj}(k)^{\mathrm{op}} & =\!=\!=\!=\!= & \mathrm{SmProj}(k)^{\mathrm{op}} \\ {\scriptstyle M(-)_{R_\mathbb{Q}}}\downarrow & & \\ \mathsf{Chow}(k)_{R_\mathbb{Q}} & & \downarrow {\scriptstyle M_0(-)_R} \\ {\scriptstyle \pi}\downarrow & & \\ \mathsf{Chow}(k)_{R_\mathbb{Q}}/_{-\otimes \mathbf{L}} & \xleftarrow[\Psi]{\qquad} & \mathsf{KM}(k)_R\,. \end{array} \tag{3.2}$$

Proof. Let X and Y be irreducible smooth projective k-schemes of dimensions d_X and d_Y. As explained by André in [1, §4], one has canonical isomorphisms

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R_\mathbb{Q}}}(M(X)_{R_\mathbb{Q}}, M(Y)_{R_\mathbb{Q}} \otimes \mathbf{L}^{\otimes j}) \simeq CH^{d_X - j}(X \times Y)_{R_\mathbb{Q}}\,,$$

where $CH^{d_X-j}(X \times Y)_{R_\mathbb{Q}}$ is the $R_\mathbb{Q}$-linear Chow group of algebraic cycles of codimension $d_X - j$ on $X \times Y$. By definition of the orbit category, the $R_\mathbb{Q}$-module

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R_\mathbb{Q}}/_{-\otimes \mathbf{L}}}(\pi(M(X)_{R_\mathbb{Q}}), \pi(M(Y)_{R_\mathbb{Q}}))$$

identifies with the direct sum $\oplus_{j\in\mathbb{Z}} CH^{d_X-j}(X \times Y)_{R_\mathbb{Q}} = CH^*(X \times Y)_{R_\mathbb{Q}}$. This shows us that the orbit category agrees with the category $\mathbf{CHM}_{kR_\mathbb{Q}}$ of *all* correspondences introduced by Gillet and Soulé in [8, page 3128]. On the other hand, recall from Section 2.2 that $\mathrm{Hom}_{\mathsf{KM}(k)_R}(M_0(X)_R, M_0(Y)_R) = K_0(X \times Y)_R$. The desired functor Ψ is defined on objects by sending $M_0(X)_R$ to $\pi(M(X)_{R_\mathbb{Q}})$. On morphisms it is defined by the following assignment

$$K_0(X \times Y)_R \longrightarrow CH^*(X \times Y)_{R_\mathbb{Q}} \qquad \alpha \mapsto ch(\alpha) \cdot \pi_Y^*(\mathrm{Td}(Y))\,,$$

where $ch(-)$ denotes the Chern character, $\mathrm{Td}(Y)$ the Todd class of Y, and π_Y the projection $X \times Y \to Y$ morphism. As explained by Gillet and Soulé in [8, page 3128], it follows from the Grothendieck–Riemann–Roch theorem that the above assignments give rise to an additive functor Ψ. The commutativity of diagram (3.2) follows also from the Grothendieck–Riemann–Roch theorem; see [8, page 3129]. □

Consider the following diagram of additive functors

$$\mathsf{Chow}(k)_R \to \cdots \to \mathsf{Chow}(k)_{R[1/n!]} \to \mathsf{Chow}(k)_{R[1/(n+1)!]} \to \cdots \to \mathsf{Chow}(k)_{R_\mathbb{Q}}\,.$$

Since the Lefschetz motive is mapped to itself, the functoriality of orbit categories gives rise to the following diagram of additive functors

$$\cdots\cdots \to \mathsf{Chow}(k)_{R[1/n!]}/_{-\otimes \mathbf{L}} \to \mathsf{Chow}(k)_{R[1/(n+1)!]}/_{-\otimes \mathbf{L}} \to \cdots \to \mathsf{Chow}(k)_{R_\mathbb{Q}}/_{-\otimes \mathbf{L}}\,.$$

Proposition 3.3. *Given irreducible smooth projective k-schemes $X_1, \ldots, X_n$ of dimension $d_1, \ldots, d_n$, the following composition (see Notation 2.1)*

$$(X_1, \ldots, X_n)_R \subset \mathsf{KM}(k)_R \xrightarrow{\Psi} \mathsf{Chow}(k)_{R_\mathbb{Q}}/_{-\otimes \mathbf{L}}$$

factors through the following functor (where $d := \max\{d_1, \ldots, d_n\}$):

$$\mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes \mathbf{L}} \longrightarrow \mathsf{Chow}(k)_{R_\mathbb{Q}}/_{-\otimes \mathbf{L}}\,.$$

Proof. Let $X_r, X_s \in \{X_1, \ldots, X_n\}$. By the very construction of the functor Ψ (see the proof of Proposition 3.1), it suffices to show that

$$K_0(X_r \times X_s)_R \longrightarrow CH^*(X_r \times X_s)_{R_{\mathbb{Q}}} \qquad \alpha \mapsto ch(\alpha) \cdot \pi^*_{X_s}(\mathrm{Td}(X_s))$$

factors through the homomorphism $CH^*(X_r \times X_s)_{R[1/(2d)!]} \to CH^*(X_r \times X_s)_{R_{\mathbb{Q}}}$. This is now standard by the Grothendieck–Riemann–Roch theorem. □

We now have all the ingredients necessary for the proof of Theorem 1.4. Recall that by hypothesis $U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \simeq \oplus_{i=1}^m \mathbf{1}$ for some integer m. Since $\mathbf{1} = U(\underline{k})_R \simeq U(\mathcal{D}^b_{\mathsf{dg}}(\mathrm{Spec}(k)))_R$, one hence concludes from diagram (2.3) and from the additivity and fully faithfulness of θ that $M_0(X)_R \simeq \oplus_{i=1}^m M_0(\mathrm{Spec}(k))_R$. Recall that the orbit category $\mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes \mathbf{L}}$ is additive. Therefore, by extending the functor Ψ to finite direct sums (see Notation 2.1), one obtains an additive functor

$$\Psi^{\oplus} : (X, \mathrm{Spec}(k))^{\oplus} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes \mathbf{L}}$$

and consequently an isomorphism

$$\pi(M(X)_{R[1/(2d)!]}) \simeq \oplus_{i=1}^m \pi(M(\mathrm{Spec}(k))_{R[1/(2d)!]}) \,. \tag{3.4}$$

Since the projection functor π is additive and $M(\mathrm{Spec}(k))_{R[1/(2d)!]}$ is the $\otimes$-unit of the category of Chow motives, there exist morphisms in the orbit category

$$\underline{f} = \{f_j\}_{j\in\mathbb{Z}} \in \oplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(M(X), \oplus_{i=1}^m \mathbf{L}^{\otimes j})$$
$$\underline{g} = \{g_j\}_{j\in\mathbb{Z}} \in \oplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\oplus_{i=1}^m M(\mathrm{Spec}(k)), M(X) \otimes \mathbf{L}^{\otimes j})$$

verifying the equalities $\underline{g} \circ \underline{f} = \mathrm{id} = \underline{f} \circ \underline{g}$; note that we have removed some subscripts in order to simplify the exposition. As explained by André in [1, §4], one has

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(M(X), \oplus_{i=1}^m \mathbf{L}^{\otimes j}) \simeq \oplus_{i=1}^m CH^{d-j}(X)_{R[1/(2d)!]}$$
$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\oplus_{i=1}^m M(\mathrm{Spec}(k)), M(X) \otimes \mathbf{L}^{\otimes j}) \simeq \oplus_{i=1}^m CH^{-j}(X)_{R[1/(2d)!]} \,.$$

As a consequence, $f_j = 0$ for $j \neq \{0, \ldots, d\}$ and $g_j = 0$ for $j \neq \{-d, \ldots, 0\}$. The sets $\{f_l \,|\, 0 \leq l \leq d\}$ and $\{g_l \otimes \mathbf{L}^{\otimes l} \,|\, -d \leq l \leq 0\}$ give then rise to morphisms

$$\alpha : M(X)_{R[1/(2d)!]} \longrightarrow \oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l} \qquad \beta : \oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l} \longrightarrow M(X)_{R[1/(2d)!]} \,.$$

The composition $\beta \circ \alpha$ agrees with the 0th-component of the composition $\underline{g} \circ \underline{f} = \mathrm{id}$, *i.e.* it agrees with the identity of $M(X)_{R[1/(2d)!]}$. We hence conclude that $M(X)_{R[1/(2d)!]}$ is a direct factor of the Chow motive $\oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l}$.

By definition of the Lefschetz motive $\mathbf{L}$, we have the following equalities

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\mathbf{L}^{\otimes p}, \mathbf{L}^{\otimes q}) = \delta_{pq} \cdot R[1/(2d)!] \qquad p, q \geq 0 \,, \tag{3.5}$$

where δ_{pq} stands for the Kronecker symbol. This implies that $M(X)_{R[1/(2d)!]}$ decomposes into a direct sum (indexed by l) of direct factors of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$. Note that a direct factor of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ is the same data as an idempotent element of $\mathrm{End}(\oplus_{i=1}^m \mathbf{L}^{\otimes l})$. Thanks to the above equality (3.5), $\mathrm{End}(\oplus_{i=1}^m \mathbf{L}^{\otimes l})$ identifies with the $m \times m$ matrices $M_{m\times m}(R[1/(2d)!])$ with coefficients in $R[1/(2d)!]$. Hence, a direct factor of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ is the same data as an idempotent element of $M_{m\times m}(R[1/(2d)!])$, *i.e.* a finitely projective $R[1/(2d)!]$-module. Since by hypothesis all these modules are free we then conclude that the only direct factors of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ are its subsums. Consequently, $M(X)_{R[1/(2d)!]}$ is isomorphic to a subsum of $\oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l}$ indexed by a subset S of $\{0, \dots, d\} \times \{1, \dots, m\}$. By construction of the orbit category, we have $\pi(\mathbf{L}^{\otimes l}) \simeq \pi(M(\mathrm{Spec}(k))_{\mathbb{Z}[1/(2d)!]})$ for every $l \geq 0$. Therefore, since the above direct sum (3.4) contains m terms, we conclude that the cardinality of S is also m. This means that there is a choice of integers (up to permutation) $l_1, \dots, l_m \in \{0, \dots, d\}$ giving rise to the desired isomorphism (1.5).

4. Proof of Proposition 1.7

Note that since the quadratic form q is anisotropic, the associated quadric Q_q has no rational points. This implies that $M(Q_q)_{\mathbb{Z}}$ cannot be of Lefschetz type. Let us now prove item (i). Recall from Kapranov [10] that we have a semi-orthogonal decomposition $\mathcal{D}^b(Q_q) = \langle \mathcal{D}^b(\mathcal{C}_0(q)), \mathcal{O}(-d+1), \dots, \mathcal{O} \rangle$, where $\mathcal{C}_0(q)$ denotes the even Clifford algebra of q and d the dimension of Q_q. Recall from Section 1 that $\underline{\mathcal{C}_0(q)}$ stands for the dg category with a single object and (dg) k-algebra of endomorphisms $\mathcal{C}_0(q)$. As proved in [19, §5], semi-orthogonal decompositions become direct sums in the category of noncommutative motives. Since $\mathcal{D}^b_{\mathsf{dg}}(\mathcal{C}_0(q))$ is Morita equivalent to $\underline{\mathcal{C}_0(q)}$, one hence obtains the following motivic decomposition

$$U(\mathcal{D}^b_{\mathsf{dg}}(Q_q))_{\mathbb{Z}} \simeq U(\underline{\mathcal{C}_0(q)})_{\mathbb{Z}} \oplus \mathbf{1}^{\oplus d}. \tag{4.1}$$

By assumption, q is even dimensional, anisotropic, and has trivial discriminant and trivial Clifford invariant. It follows then from Lam [17, page 111] that the even Clifford algebra $\mathcal{C}_0(q)$ is isomorphic to $M_r(k) \times M_r(k)$ where $r := 2^d$ and $M_r(k)$ is the algebra of $r \times r$ matrices over k. In particular, $\mathcal{C}_0$ is Morita equivalent to $k \times k$. This allows us to conclude that the right-hand side of (4.1) identifies with $\mathbf{1}^{\oplus(d+2)}$ and consequently that $U(\mathcal{D}^b_{\mathsf{dg}}(Q_q))_{\mathbb{Z}}$ is of trivial type.

5. Proof of Theorem 1.11

The following result is well-known; see Soulé [22] for example.

Proposition 5.1. *Let X be an irreducible smooth projective k-scheme of dimension d_X. Under the assumption $\mathbb{Z} \subseteq R$, the following holds:*

(i) *The Todd class* $\mathrm{Td}(X)$ *is invertible in the Chow ring* $CH^*(X)_{R[1/(2d_X)!]}$.
(ii) *We have a Chern character isomorphism* $K_0(X)_{R[1/(2d_X)!]} \simeq CH^*(X)_{R[1/(2d_X)!]}$.

Let $X_1, \dots, X_n$ be irreducible smooth projective k-schemes of dimensions $d_1, \dots, d_n$. Recall from Section 3 the construction of the functor (where $d := \max\{d_1, \dots, d_n\}$):

$$\Psi^{\oplus} : (X_1, \dots, X_n)^{\oplus}_R \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes \mathbf{L}}.$$

Proposition 5.2. *The induced $R[1/(2d)!]$-linear functor is fully faithful*

$$\Psi^{\oplus} : (X_1, \dots, X_n)^{\oplus}_{R[1/(2d)!]} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes \mathbf{L}}.$$

Proof. Let $X_r, X_s \in \{X_1, \dots, X_n\}$. By the very construction of the functor $\Psi^{\oplus}$, it suffices to show that the following homomorphism is invertible

$$K_0(X_r \times X_s)_{R[1/(2d)!]} \longrightarrow CH^*(X_r \times X_s)_{R[1/(2d)!]} \qquad \alpha \mapsto ch(\alpha) \cdot \pi^*_{X_s}(\mathrm{Td}(X_s)).$$

This follows automatically from Proposition 5.1 above. □

Recall that by hypothesis X is an irreducible smooth projective k-scheme of dimension d. By combining the commutativity of diagram (2.3) with the fully faithfulness of the functor θ one obtains an $R[1/(2d)!]$-algebra isomorphism

$$\mathrm{End}(U(\mathcal{D}^b_{\mathsf{dg}}(X))_{R[1/(2d)!]}) \simeq \mathrm{End}(M_0(X)_{R[1/(2d)!]}) \,.$$

Thanks to the fully faithfulness of the functor $\Psi^\oplus$ of Proposition 5.2 (applied to the category $(X)^\oplus_{R[1/(2d)!]}$) and the commutativity of diagram (3.2), one has moreover

$$\mathrm{End}(M_0(X)_{R[1/(2d)!]}) \simeq \mathrm{End}(\Psi^\oplus(M_0(X)_{R[1/(2d)!]})) \simeq \mathrm{End}(\pi(M(X)_{R[1/(2d)!]})) \,.$$

Since projection functor π is faithful, one hence obtains the following inclusion

$$\mathrm{End}(M(X)_{R[1/(2d)!]}) \hookrightarrow \mathrm{End}(U(\mathcal{D}^b_{\mathsf{dg}}(X))_{R[1/(2d)!]}) \,.$$

This automatically gives rise to the desired implication (1.12).

6. Proof of Proposition 1.13

Item (ii) was proved by Karpenko, see [11, Thm. 2.2.1] for the first statement and [12, Cor. 2.22] for the second one. Let us now prove item (i). Recall from [3] that we have the semi-orthogonal decomposition

$$\mathcal{D}^b(X) = \langle \mathcal{D}^b(k), \mathcal{D}^b(A), \mathcal{D}^b(A^{\otimes 2}), \ldots, \mathcal{D}^b(A^{\otimes d-1}) \rangle \,.$$

As proved in [19, §5], semi-orthogonal decompositions become direct sums in the category of noncommutative motives. Since $\mathcal{D}^b_{\mathsf{dg}}(A^{\otimes i})$ is Morita equivalent to $\underline{A}^{\otimes i}$, one hence obtains the following motivic decomposition

$$U(\mathcal{D}^b_{\mathsf{dg}}(X))_R \simeq U(\underline{k})_R \oplus U(\underline{A})_R \oplus U(\underline{A}^{\otimes 2})_R \oplus \ldots \oplus U(\underline{A}^{\otimes d-1})_R \,. \tag{6.1}$$

Finally, since the functor $U(-)_R$ is symmetric monoidal, (6.1) identifies with (1.14).

7. Proof of Theorem 1.16

Note first that, by combining the commutativity of diagram (2.3) with the fully faithfulness of the functor θ, it suffices to prove the implication

$$\oplus_{i=1}^n M(X_i)_R \otimes \mathbf{L}^{\otimes l_i} \simeq \oplus_{j=1}^m M(Y_j)_R \otimes \mathbf{L}^{\otimes l_j} \Rightarrow \oplus_{i=1}^n M_0(X_i)_R \simeq \oplus_{j=1}^m M_0(Y_j)_R \,.$$

As mentioned in Section 2.3, the projection functor π is additive and sends $M(X_i)_R \otimes \mathbf{L}^{\otimes l_i}$ to $\pi(M(X_i)_R)$ (up to isomorphism). Hence, the left-hand side of the above implication gives rise to an isomorphism $\oplus_{i=1}^n \pi(M(X_i)_R) \simeq \oplus_{j=1}^m \pi(M(Y_j)_R)$. Since by hypothesis $1/(2d)! \in R$, Proposition 5.2 gives us a fully faithful functor

$$\Psi^\oplus : (X_1, \ldots, X_n, Y_1, \ldots, Y_m)^\oplus_R \longrightarrow \mathsf{Chow}(k)_R/_{-\otimes \mathbf{L}} \,.$$

Using this functor and the commutativity of diagram (3.2), one observes that

$$\Psi^\oplus(\oplus_{i=1}^n M_0(X_i)_R) \simeq \oplus_{i=1}^n \pi(M(X_i))_R \quad \Psi^\oplus(\oplus_{j=1}^m M_0(Y_j)_R) \simeq \oplus_{j=1}^m \pi(M(Y_j))_R \,.$$

Finally, by combining all the above isomorphisms, one hence obtains the right-hand side of the above implication, which concludes the proof.

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