The distribution of the product of powers of independent Uniform random variables
a simple but useful tool to address and better understand the structure of some distributions

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Abstract

We begin with some simple questions. What is the distribution of the product of given powers of independent continuous uniform random variables? Is this distribution useful? Are there some test statistics with this distribution? Is this distribution somehow related to the distribution of the product of other random variables? Is this distribution already known in some context? This short paper will give the answers to the above questions. It will be seen that the answer to the last four questions above is: yes! We will show how this distribution may help us to shed some new light on other well known distributions and also how it may help us in approaching in a much simpler way some distributions usually considered to be rather complicated.

Key words: product of independent Beta random variables, Wilks Lambda statistic, Generalized Integer Gamma distribution.

1 Introduction

This paper will provide a different view of some rather well known distributions and some relations among them. Such new insights, besides shedding some new light on these distributions may also help us understand the problems we face when trying to handle closely related distributions which are far less manageable than the ones studied in this paper and the reasons why that happens.
The distributions we are interested in studying are the distributions of r.v.’s (random variables) of the form

\[ V = \prod_{j=1}^{p} \prod_{i=1}^{r_j} \frac{U_{ij}^{1}}{\delta_{ij}}, \quad \text{for } \delta_{ij} > 0, \quad (1) \]

where the \( U_{ij} \)'s are i.i.d. \( Unif(0,1) \) r.v.’s, for \( j = 1, \ldots, p; \ i = 1, \ldots, r_j \).

In terms of notation, we will use \( \mathbb{N} \) to denote the set of positive integers, \( \mathbb{R} \) to denote the set of reals, \( \mathbb{R}^+ \) to denote the set of positive reals and \( \mathbb{C} \) to denote the set of complex numbers. We will also use “\( \sim \)” to denote “is distributed as” and “\( \overset{d}{\sim} \)” to denote “has the same distribution as” or “is stochastically equivalent to”.

2 Several distributions which may be represented as the distribution of the product of powers of independent Uniform r.v.’s

In this section we will show how particular choices of \( \delta_{ij} \) will lead to \( V \) in (1) having interesting and useful distributions, whose p.d.f.’s (probability density functions) and c.d.f.’s (cumulative distribution functions) have concise and manageable representations.

2.1 The general case – the product of powers of Uniforms as an exponentiated generalized integer gamma (EGIG) distribution and as the distribution of the product of Pareto r.v.’s

If all the \( \delta_{ij} \)'s are distinct, \( V \) in (1) has an exponentiated generalized integer gamma (EGIG) distribution of depth \( \sum_{j=1}^{p} r_j \) (see Appendix B) with all shape parameters equal to 1 and rate parameters \( \delta_{ij} \), with p.d.f. and c.d.f. respectively given by

\[ f_V(v) = \left( \prod_{j=1}^{p} \prod_{i=1}^{r_j} \delta_{ij} \right) \sum_{j=1}^{p} \sum_{i=1}^{r_j} c_{ij} v^{\delta_{ij} - 1} \]

and

\[ F_V(v) = \left( \prod_{j=1}^{p} \prod_{i=1}^{r_j} \delta_{ij} \right) \sum_{j=1}^{p} \sum_{i=1}^{r_j} c_{ij} \frac{v^{\delta_{ij}}}{\delta_{ij}} \]
with
\[ c_{ij} = \prod_{h=1}^{p} \prod_{\ell=1}^{r_h} \frac{1}{\delta_{th} - \delta_{ij}}. \]

For \( \delta_{ij} = \delta_j \) for all \( i = 1, \ldots, r_j \), the distribution of \( V \) is an EGIG distribution of depth \( p \) with shape parameters \( r_j \) and rate parameters \( \delta_j \), assuming that all the \( \delta_j \)'s are different for \( j \in \{1, \ldots, p\} \). In case there are some equalities among the \( \delta_j \)'s, we only have to group the corresponding \( U_{ij} \)'s in (1) or (2) under the same product in \( i \), this way adding the corresponding \( r_j \)'s and thus reducing the depth of the distribution, which will be equal to the number of different \( \delta_j \)'s.

The distribution of
\[ V_j = \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{\delta_j}} \]
where the \( U_{ij} \)'s are i.i.d. \( Unif(0, 1) \) r.v.'s, is a simple one, since, using the relation in section A.1 in Appendix A, we know that
\[-\log U_{ij}^{\frac{1}{\delta_j}} \sim Exp(\delta_j),\]
so that
\[-\log V_j = \sum_{i=1}^{r_j} -\log U_{ij}^{\frac{1}{\delta_j}} \sim \Gamma(r_j, \delta_j),\]
that is, \(-\log V_j\) is an r.v. with p.d.f.
\[ f_{-\log V_j}(w) = \frac{\delta_j^{r_j}}{(r_j - 1)!} e^{-\delta_j w} w^{r_j - 1}, \quad (w > 0) \]
so that \( V_j \) has p.d.f.
\[ f_{V_j}(v) = \frac{\delta_j^{r_j}}{(r_j - 1)!} v^{\delta_j - 1} (-\log v)^{r_j - 1}, \quad 0 < v < 1. \]

Of course, if \( \delta_j = 1 \), this reduces to the distribution of the product of \( r_j \) i.i.d. \( Unif(0, 1) \) r.v.'s, with p.d.f.
\[ f_{V_j}(v) = \frac{1}{(r_j - 1)!} (-\log v)^{r_j - 1}, \quad 0 < v < 1. \]
The c.d.f. of $V_j$ is then easy to obtain.

If in (1) we take $r_j = 1$ and $\delta_{ij} = \delta_j < 0$ ($j = 1, \ldots, p$), the r.v.’s $\theta_j U_{ij}^{{\frac{1}{\delta_j}}}$ will have $\text{Pareto}(\theta_j, \delta_j)$ distributions (see Appendix A). Thus, in this case

$$V = \prod_{j=1}^{p} \theta_j U_{ij}^{{\frac{1}{\delta_j}}}$$

will have the distribution of the product of $p$ independent r.v.’s with $\text{Pareto}(\theta_j, \delta_j)$ distributions.

But then, the r.v.

$$V^* = \left( \prod_{j=1}^{p} \theta_j \right) V^{-1} = \prod_{j=1}^{p} U_{ij}^{-{\frac{1}{\delta_j}}}$$

will have an EGIG distribution of depth $p$ (if all $\delta_j$’s are different), with all shape parameters equal to 1 and rate parameters $(-\delta_j)$, so that $V$ will have p.d.f. and c.d.f. respectively given by (see Appendix B),

$$f_V(v) = f_{\text{EGIG}} \left( \left( \prod_{j=1}^{p} \theta_j \right) v^{-1}; 1, -\delta_j; p \right) \frac{1}{v^2} \prod_{j=1}^{p} \theta_j$$

$$= f_{\text{GIG}} \left( \log v - \sum_{j=1}^{p} \log \theta_j; 1, -\delta_j; p \right) \frac{1}{v^3} \prod_{j=1}^{p} \theta_j , \quad (1 < v < \infty)$$

and

$$F_V(v) = F_{\text{EGIG}} \left( \left( \prod_{j=1}^{p} \theta_j \right) v^{-1}; 1, -\delta_j; p \right)$$

$$= 1 - F_{\text{GIG}} \left( \log v - \sum_{j=1}^{p} \log \theta_j; 1, -\delta_j; p \right) , \quad (1 < v < \infty) .$$

We may note that in this case we have

$$E \left( V^h \right) = \prod_{j=1}^{p} \theta_j^h \frac{-\delta_j}{-\delta_j - h} \left( h < - \max_{1 \leq i \leq p} \delta_j \right) .$$

If some of the $\delta_j$’s are equal, we only have to group the corresponding $U_{ij}$’s under the same product in $i$, this way reducing the depth of the EGIG or
GIG distribution, which will then be equal to the number of different \( \delta_j \)'s. At the same time we will have to increase the corresponding shape parameters so that they reflect for each \( \delta_j \) the number of times it appears repeated. More precisely, in this case \( V^* \) would be written as

\[
V^* = \prod_{j=1}^{p^*} \prod_{i=1}^{r_j} U_{ij}^{-\frac{1}{\delta_j}}
\]

where \( p^* \leq p \) is the number of different \( \delta_j \)'s and as such is the depth of the EGIG distribution, and \( r_j \) is the number of times that \( \delta_j \) appears repeated. Then, in this case the \( r_j \)'s will be the shape parameters in the EGIG distribution of \( V \), which will have p.d.f. and c.d.f. respectively given by

\[
f_V(v) = f_{\text{EGIG}} \left( \left( \prod_{j=1}^{p} \theta_j \right) v^{-1}; r_j, -\delta_j; p^* \right) \frac{1}{v^{\frac{1}{2}}} \prod_{j=1}^{p} \theta_j
\]

\[
= f_{\text{GIG}} \left( \log v - \sum_{j=1}^{p} \log \theta_j; r_j, -\delta_j; p^* \right) \frac{1}{v^{\frac{3}{2}}} \prod_{j=1}^{p} \theta_j, \quad (1 < v < \infty)
\]

and

\[
F_V(v) = F_{\text{EGIG}} \left( \left( \prod_{j=1}^{p} \theta_j \right) v^{-1}; r_j, -\delta_j; p^* \right)
\]

\[
= 1 - F_{\text{GIG}} \left( \log v - \sum_{j=1}^{p} \log \theta_j; r_j, -\delta_j; p^* \right), \quad (1 < v < \infty).
\]

Note that these results agree with those given in Arnold (1983) for the cases in which the \( \delta_j \)'s are either all equal or all different.

2.2 When the product of powers of Uniforms is a Beta or a product of Betas

By looking at the results in section A.2 in Appendix A, we may easily see that if in (2) we take \( \delta_{ij} = a_j + i - 1 \) \((a_j > 0)\), then,

\[
V_j = \prod_{i=1}^{r_j} U_{ij}^{-\frac{1}{\delta_{ij}+1-i}} \sim \text{Beta}(a_j, r_j)
\]
and as such, in this case,

\[ V = \prod_{j=1}^{p} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{r_j} + \frac{i}{k} - \frac{1}{k}} \sim \prod_{j=1}^{p} Y_j, \quad \text{where } Y_j \sim \text{Beta}(a_j, r_j) \]

are \( p \) independent r.v.'s \( (r_j \in \mathbb{N}) \) \( (3) \)

Further, from section A.3 in Appendix A, if in (1) we take \( \delta_{ij} = a_j + i/k - 1/k \), we will have

\[ V = \prod_{j=1}^{p} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{r_j} + \frac{i}{k} - \frac{1}{k}} \sim \prod_{j=1}^{p} \prod_{i=1}^{k} Y_{ji}, \quad \text{where } Y_{ji} \sim \text{Beta} \left( a_j + \frac{i-1}{k}, \frac{r_j}{k} \right) \]

are all independent r.v.'s. \( (4) \)

If all the \( a_j + i - 1 \) in (3) and all the \( a_j + i/2 - 1/2 \) in (4) are different for \( j = 1, \ldots, p, i = 1, \ldots, r_j \), the distribution of \( V \) in both cases will be an EGIG distribution of depth \( \sum_{j=1}^{p} r_j \), with shape parameters all equal to 1 and rate parameters \( a_j + i - 1 \) in the first case and \( a_j + i/2 - 1/2 \) in the second case, with p.d.f. and c.d.f. given respectively by (B.3) and (B.4) in Appendix B. In case some of the \( a_j + i - 1 \) in (3) or some of the \( a_j + i/2 - 1/2 \) in (4) are equal, then the depth of the EGIG distribution has to be reduced accordingly and the corresponding shape parameters increased accordingly.

As a particular case of the distribution in (4), we have, the distribution, for even \( p \), of

\[ \prod_{j=1}^{p} Y_j, \quad \text{where } Y_j \sim \text{Beta} \left( a + \frac{i}{2}, \frac{b}{2} \right), \quad (a > 0, b \in \mathbb{N}) \]

are \( p \) independent r.v.’s \( (5) \)

which is then the distribution of

\[ V = \prod_{j=1}^{p/2} \prod_{i=1}^{b} U_{ij}^{\frac{1}{a + \frac{i}{2} - \frac{1}{2}} + \frac{b}{2} - \frac{1}{2}} \]

\( (6) \)

where \( a + \frac{2j-1}{2} + \frac{i}{2} - \frac{1}{2} \) varies between \( a + \frac{1}{2} \) and \( a + \frac{b}{2} + 1 \), with \( a + \frac{i}{2} \) occurring \( r_j \) times \( (j = 1, \ldots, p + b - 2) \), with

\[ r_j = \begin{cases} h_j, & j = 1, 2 \\ h_j + r_{j-2}, & j = 3, \ldots, p + b - 2 \end{cases} \]

\( (7) \)
where
\[ h_j = \begin{cases} 
1, & j = 1, \ldots, \min(p, b) \\
0, & j = 1 + \min(p, b), \ldots, \max(p, b) \\
-1, & j = 1 + \max(p, b), \ldots, p + b - 2,
\end{cases} \quad (8) \]
so that we may write
\[ V = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r} U_{ij}^{\frac{1}{2}}, \quad (9) \]
which shows that the exact distribution of \( V \) is in this case an EGIG distribution of depth \( p + b - 2 \) with shape parameters \( r_j \) and rate parameters \( a + \frac{j}{2} \) \((j = 1, \ldots, p + b - 2)\).

It is interesting to note that the representation for \( V \) in (9) clearly indicates the role of the parameters \( a + \frac{j}{2} \) as the powers of the \( Unif(0, 1) \) r.v.’s, corresponding to the \( p + b - 2 \) first parameters of the Beta r.v.’s appearing in the product-of-betas representation of \( V \).

We may also note that just by writing the first parameters of the Beta r.v.’s in (5) in reverse order, we may see that we have for \( V \) in (9),
\[ V \sim \prod_{j=1}^{p} Y_j, \quad \text{where } Y_j \sim Beta \left( a + \frac{p+1}{2} - \frac{j}{2}, \frac{b}{2} \right) \]
are independent r.v.’s,
so that, given the symmetry of the \( r_j \)’s in (7), with \( r_j = r_{p+b-1-j} \), we have for \( V \) in (9),
\[ V = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r} U_{ij}^{\frac{1}{2}} = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r} U_{ij}^{\frac{p+b-1-j}{2}}. \]
For ease of notation for the \( r_j \)’s and the \( \delta_j \)’s, in the next section we will actually prefer the second of these representations..

### 2.3 The exact distribution of the Wilks \( \Lambda \) statistic

If we assume
\[ X = [X'_1, X'_2, \ldots, X'_m]' \sim N_p(\mu, \Sigma) \]
where

\[ \mu = [\mu_1', \mu_2', \ldots, \mu_m']' \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1m} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m1} & \Sigma_{m2} & \cdots & \Sigma_{mm} \end{bmatrix} \]

with

\[ \mu_k = E(X_k) \quad \text{and} \quad \Sigma_{kk} = Var(X_k), \quad \Sigma_{kk'} = Cov(X_k, X_{k'}), \quad k, k' \in \{1, \ldots, m\}, \]

and we wish to test the null hypothesis

\[ H_0 : \Sigma_{kk'} = 0 \quad \text{for all} \quad k \neq k' \in \{1, \ldots, m\}, \quad (10) \]

that is the null hypothesis of independence of the \( m \) sets of variables \( X_k \) \((k = 1, \ldots, m)\), the l.r.t. (likelihood ratio test) statistic is

\[ \Lambda = \frac{|S|}{\prod_{k=1}^{m} |S_{kk}|} \]

where \( S \) is the MLE of \( \Sigma \) and \( S_{kk} \) the MLE of \( \Sigma_{kk} \) \((k = 1, \ldots, m)\).

Let us suppose that we have a sample of size \( n + 1 \) from the distribution of \( X \) and suppose that each \( X_k \) has \( p_k \) variables and let us suppose that at most one of these \( p_k \)'s is odd. Without any loss of generality, let it be \( p_m \).

For \( m = 2 \) we know that, taking \( p = p_1 + p_2 \),

\[ \Lambda \overset{\text{st}}{= \prod_{j=1}^{p_1} Y_j, \quad \text{with} \quad Y_j \sim Beta\left(\frac{n-p+j}{2}, \frac{p_2}{2}\right)} \]

or \( Y_j \sim Beta\left(\frac{n-1-j}{2}, \frac{p_2}{2}\right) \), all independent

so that we may write, from (5) and (9), for \( r_j \) defined as in (7) and (8), with \( p \) and \( b \) replaced respectively by \( p_1 \) and \( p_2 \),

\[ \Lambda = \prod_{j=1}^{p_1+p_2-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{(n-p+j)/2}} = \prod_{j=1}^{p_1+p_2-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{(n-1-j)/2}}, \]

which shows that the exact distribution of \( \Lambda \) is, in this case, an EGIG distribution of depth \( p_1 + p_2 - 2 \), with shape parameters \( r_j \), given by (7) and (8),
and rate parameters $\frac{n-1-j}{2} (j = 1, \ldots, p_1 + p_2 - 2)$, with p.d.f. and c.d.f. given by (B.3) and (B.4) in Appendix B.

For $m > 2$, it may be shown (Anderson, 2003), that

$$\Lambda = \prod_{k=1}^{m-1} \Lambda_{k,(k+1,\ldots,m)}, \quad (11)$$

where $\Lambda_{k,(k+1,\ldots,m)} (k = 1, \ldots, m-1)$ is the l.r.t. statistic to test

$$H_{0k} : \Sigma_{kl} = 0, \ell = k + 1, \ldots, m \quad (k = 1, \ldots, m-1).$$

The $\Lambda_{k,(k+1,\ldots,m)}$’s are independent under $H_0$ in (10), and they may be represented in the form

$$\Lambda_{k,(k+1,\ldots,m)} \sim \prod_{j=1}^{p_k} Y_j, \quad \text{with } Y_j \sim \text{Beta}\left(\frac{n+1-q_k-j}{2}, \frac{q_k}{2}\right) \quad \text{or } Y_j \sim \text{Beta}\left(\frac{n-q_k-1+j}{2}, \frac{q_k}{2}\right), \quad \text{all independent} \quad (12)$$

where $q_k = p_{k+1} + \ldots + p_m$, so that, from (5) and (6) and then (9),

$$\Lambda = \prod_{k=1}^{m-1} \prod_{j=1}^{p_k} Y_j = \prod_{k=1}^{m-1} \left( \prod_{j=1}^{p_k/2} q_k \prod_{i=1}^{n-q_k-j+1} U_{ij} \right)^{\frac{1}{2}} \prod_{i=1}^{n-q_k-j+1} U_{ij}^{\frac{1}{2}} \quad (13)$$

where the $r_{kj}$, for $k = 1, \ldots, m-1$, are defined in the same manner as the $r_j$’s in (7) and (8) with $p$ and $b$ replaced respectively by $p_k$ and $q_k$. But it is evident that in (13) there are several repetitions of each $\delta_{kj} = \frac{n-q_k+j}{2}$, which indeed, for $p = p_1 + \ldots + p_m$, range from a minimum value of $\frac{n-p+1}{2}$ through a maximum of $\frac{n-2}{2}$ with $\delta_j = \frac{n-1-j}{2} (j=1, \ldots, p-2)$ occurring $r_j$ times, where

$$r_j = \sum_{k=1}^{m-1} r_{kj} \quad (14)$$

in which, for $k = 1, \ldots, m-1$,

$$r_{kj} = \begin{cases} h_{kj} & j = 1, 2 \\ h_{kj} + r_{k,j-2} & j = 3, \ldots, p_k + q_k - 2 \\ 0 & j = p_k + q_k - 1, \ldots, p - 2 \end{cases} \quad (15)$$
with \( h_{kj} \) \((k = 1, \ldots, m - 1; j = 1, \ldots, p_k + q_k - 2)\) given by (8) with \( p \) and \( b \) replaced respectively by \( p_k \) and \( q_k \). Consequently we may write

\[
\Lambda = \prod_{j=1}^{p-2} \prod_{i=1}^{r_j} U_{ij}^{(n-j)/2} \tag{16}
\]

which shows that the exact distribution of \( \Lambda \) is in this case an EGIG distribution of depth \( p-2 \) with shape parameters \( r_j \) and rate parameters \( \delta_j = \frac{n-1-j}{2} \) \((j = 1, \ldots, p - 2)\), with p.d.f. and c.d.f. given by (B.3) and (B.4) in Appendix B.

Note that this distribution, under \( H_0 \) in (10) is still valid if we assume for \( X \) an underlying elliptically contoured or left orthogonal-invariant distributions (Anderson et al., 1986; Anderson and Fang, 1990; Jensen and Good, 1981; Kariya, 1981).

For the general case, \( m \geq 2 \), from (11) and (12) and the expression for the \( h \)-th moment of a Beta r.v., we have the usual expression for the \( h \)-th moment of Wilks \( \Lambda \) statistic given by

\[
E(\Lambda^h) = \prod_{k=1}^{m-1} \prod_{j=1}^{p_k} \frac{\Gamma\left(\frac{n+1-j}{2}\right) \Gamma\left(\frac{n+1-q_k-1-j}{2} + h\right)}{\Gamma\left(\frac{n+1-q_k-1-j}{2} + h\right) \Gamma\left(\frac{n+1-j}{2} + h\right)}, \quad (h > -\frac{n+1-p}{2}).
\]

while from (16) and the results in Appendix A, we have for the case in which at most one of the \( m \) sets of variables has an odd number of variables, the \( h \)-th moment of \( \Lambda \) can be expressed explicitly as (see also Appendix B for the moments of the EGIG distribution)

\[
E(\Lambda^h) = \prod_{j=1}^{p-2} r_j \prod_{i=1}^{\frac{1}{n-1-j}} \frac{1}{\frac{2}{h+1}} = \prod_{j=1}^{p-2} \left(\frac{n-j-1}{n-j+2h}\right)^{r_j}, \quad (h > -\frac{n-p+1}{2})
\]

for \( r_j \) given by (14) and (15).

### 3 Conclusions

Interestingly, the distribution of \( V \) in (1) yields, for different values of the \( \delta_{ij} \)'s, several other distributions, including the exact distribution of Wilks \( \Lambda \) statistic, used to test the independence of several sets of variables. The difficulty encountered in writing down a concise and manageable expression for the exact distribution of this statistic when more than one set of variables has
an odd number of variables is related with the fact that in this case we will be left with at least one Beta r.v. with a non-integer second parameter, and as such it cannot be represented as a product of powers of $Unif(0, 1)$ r.v.’s.

The distribution of the product of powers of independent $Unif(0, 1)$ r.v.’s is a useful tool to approach in an unified way the distributions of several random variables, which would be more challenging to deal with if they were not recognized as being amenable to such a representation. The distribution of the product of powers of $Unif(0, 1)$ r.v.’s appears as the common link and common elementary structure among these distributions.

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Appendices

A Some simple relations among distributions

Many of the relations presented in this Appendix are suitable to be used as simple problems in an undergraduate course in Statistics or Distribution Theory. We list them to systematize and establish some of the notation used in this paper.

A.1 The relation between Uniform, Beta and Exponential variables

It is very easy to establish, by simple transformation, that if

$$X \sim Unif(0, 1),$$

then

$$Y = X^{1/a} \sim Beta(a, 1) \quad (a > 0),$$
and
\[ W = - \log Y = - \frac{1}{a} \log X \sim \text{Exp}(a), \]

and vice-versa, that is, if
\[ W \sim \text{Exp}(a), \]

then
\[ Y = e^{-W} \sim \text{Beta}(a, 1) \]

and
\[ X = Y^a = e^{-aW} \sim \text{Unif}(0, 1). \]

Also, if we take \( \theta > 0 \),
\[ Z = \theta Y^{-1} = \theta X^{-1/a} \sim \text{Pareto}(\theta, a) , \]

that is, \( Z \) will have p.d.f.
\[ f_Z(z) = a \frac{\theta^a}{z^{a+1}} \quad (z > \theta). \]

Concerning the moments of these r.v.’s, we may easily see that
\[ E(X^h) = \frac{1}{h+1}, \quad (h > -1) \]
\[ \implies E(Y^h) = E \left( X^{h/a} \right) = E \left( e^{-hW} \right) = \frac{a}{a+h}, \quad (h > -a) \]
\[ \implies E(Z^h) = \theta^h E(Y^{-h}) = \theta^h E(X^{-h/a}) = \theta^h \frac{a}{a-h}, \quad (h < a) \]

and
\[ E(W^h) = \frac{1}{a^h} E \left[ (-\log X)^h \right] = \frac{\Gamma(h+1)}{a^h}, \quad (h > -1), \]

which for \( h \in \mathbb{N} \) is also
\[ E(W^h) = \frac{d^h}{dt^h} E(Y^{-t}) \bigg|_{t=0} = \frac{d^h}{dt^h} \frac{a}{a-t} \bigg|_{t=0} = \frac{h!}{a^h}. \]
A.2 Some Beta r.v.'s as finite products of independent Beta r.v.'s

The following well known result is easy to establish using the representation of beta random variables in terms of independent gamma variables.

If

\[ Y \sim \text{Beta}(a, b), \ a \in \mathbb{R}^+, \ b \in \mathbb{N}, \]

then

\[ Y \overset{st}{\sim} \prod_{j=1}^{b} Y_j, \]

where \( Y_j \sim \text{Beta}(a + j - 1, 1) \) are \( b \) independent r.v.'s.

Alternatively the result can be obtained by using the mgf's or cf's of the logarithms of the random variables in question, by observing that

\[ E(Y^h) = \frac{\Gamma(a + b)}{\Gamma(a)} \frac{\Gamma(a + h)}{\Gamma(a + b + h)}, \ h > -a, \]

so that the c.f. of the r.v. \( W = -\log Y \) is given by

\[ \Phi_W(t) = \frac{\Gamma(a + b)}{\Gamma(a)} \frac{\Gamma(a - it)}{\Gamma(a + b - it)}, \ t \in \mathbb{R}. \]

Thus, using the relation

\[ \frac{\Gamma(a + n)}{\Gamma(a)} = \prod_{i=0}^{n-1} (a + i), \ a \in \mathbb{C}, \ b \in \mathbb{N}, \]

we may write

\[ \Phi_W(t) = \prod_{j=1}^{b} (a + j - 1)(a + j - i - it)^{-1}, \]

which shows that

\[ W \overset{st}{\sim} \sum_{j=1}^{b} W_j, \quad \text{where } W_j \sim \text{Exp}(a + j - 1) \]

are \( b \) independent r.v.'s.
But then, using the relation in the previous section of this Appendix,

\[ Y = e^{-W} \prod_{j=1}^{b} Y_j \]

where

\[ Y_j = e^{-W_j} \sim \text{Beta}(a + j - 1, 1) \]

are \( b \) independent r.v.'s.

Further, if

\[ Y_j \sim \text{Beta} \left( a + \frac{j - 1}{k}, \frac{b}{k} \right), \quad (j = 1, \ldots, k) \]

are \( k \) independent r.v.'s and if we define

\[ Y = \prod_{j=1}^{k} Y_j \]

we will have

\[ Y^{1/k} \sim \text{Beta}(ka, b). \]

To verify this we may use the Gamma function multiplication formula,

\[ \Gamma(kz) = (2\pi)^{\frac{1}{2}(1-k)} k^{kz-\frac{1}{2}} \prod_{j=0}^{k-1} \Gamma \left( z + \frac{j}{k} \right), \]

to write the c.f. of \(-\log Y\) as

\[
\Phi_{-\log Y}(t) = \prod_{j=1}^{k} \Phi_{-\log Y_j}(t) \\
= \prod_{j=1}^{k} \frac{\Gamma \left( a + \frac{j-1}{k} + \frac{b}{k} \right) \Gamma \left( a + \frac{j-1}{k} - it \right)}{\Gamma \left( a + \frac{j-1}{k} \right) \Gamma \left( a + \frac{j-1}{k} + \frac{b}{k} - it \right)} \\
= \frac{\Gamma(ka + b) \Gamma(ka - kit)}{\Gamma(ka) \Gamma(ka + b - kit)}
\]
so that $Y$ has the same distribution as the $k$-th power of a Beta r.v. with parameters $ka$ and $b$. Consequently we may write, for $b \in \mathbb{N}$,

$$Y^{1/k} \overset{st}{\sim} \prod_{j=1}^{b} U_j^{\frac{1}{ka+j-1}}$$

where $U_j$ are i.i.d. $Unif(0, 1)$ r.v.’s, or

$$Y \overset{st}{\sim} \prod_{j=1}^{b} U_j^{\frac{1}{a+j}}.$$ 

\textbf{B The GIG and EGIG distributions}

We will say that the r.v. $W$ has a GIG (Generalized Integer Gamma) distribution of depth $p$, with shape parameters $r_j \in \mathbb{N}$ and rate parameters $\delta_j (j = 1, \ldots, p)$, if

$$W = \sum_{j=1}^{p} W_j$$

where

$$W_j \sim \Gamma(r_j, \delta_j), \quad r_j \in \mathbb{N}, \quad \delta_j > 0, \quad j = 1, \ldots, p$$

are $p$ independent r.v.’s, and $\delta_j \neq \delta_{j'}$ for all $j, j' \in \{1, \ldots, p\}$.

The exact p.d.f. and c.d.f. of $W$ are given by (Coelho, 1998),

$$f_{GIG}(w; r_j, \delta_j; p) = K \sum_{j=1}^{p} P_j(w) \ e^{-\delta_j w}, \quad (w > 0) \quad (B.1)$$

and

$$F_{GIG}(w; r_j, \delta_j; p) = 1 - K \sum_{j=1}^{p} P_j^*(w) \ e^{-\delta_j w}, \quad (w > 0) \quad (B.2)$$

where $K = \prod_{j=1}^{p} \delta_j^{r_j}$,

$$P_j(w) = \sum_{k=1}^{r_j} c_{j,k} w^{k-1} \quad \text{and} \quad P_j^*(w) = \sum_{k=1}^{r_j} c_{j,k} (k - 1)! \sum_{i=0}^{k-1} \frac{w^i}{i! \delta_j^{k-i}}$$

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with
\[ c_{j,r} = \frac{1}{(r_j - 1)!} \prod_{i=1 \atop i \neq j}^p (\delta_i - \delta_j)^{-r_i}, \quad j = 1, \ldots, p, \]
and, for \( k = 1, \ldots, r_j - 1; j = 1, \ldots, p, \)
\[ c_{j,r_j - k} = \frac{1}{k} \sum_{i=1}^k \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!} R(i, j, p) c_{j,r_j -(k-i)}, \]
where
\[ R(i, j, p) = \sum_{k=1 \atop k \neq j}^p r_k (\delta_j - \delta_k)^{-i} \quad (i = 1, \ldots, r_j - 1). \]

We will then say that the r.v. \( V = e^{-W} \) has an EGIG (Exponentiated Generalized Integer Gamma) distribution of depth \( p \), with shape parameters \( r_j \) and rate parameters \( \delta_j \), with p.d.f. and c.d.f. respectively given by
\[
\begin{align*}
f^{\text{EGIG}}(v; r_j, \delta_j; p) &= f^{\text{GIG}}(-\log v; r_j, \delta_j; p) \frac{1}{v} \quad (0 < v < 1) \quad (B.3) \\
F^{\text{EGIG}}(v; r_j, \delta_j; p) &= 1 - F^{\text{GIG}}(-\log v; r_j, \delta_j; p) \quad (0 < v < 1). \quad (B.4)
\end{align*}
\]

Note that the \( h \)-th moment of \( V \) is, for \( h > -\min_{1 \leq j \leq p} \delta_j \), given by
\[
E(V^h) = E \left( e^{-hW} \right) = \prod_{j=1}^p E \left( e^{-hW_j} \right) = \prod_{j=1}^p \delta_j^{-r_j} (\delta_j + h)^{-r_j},
\]
while the \( h \)-th moment of \( W \), for \( h \in \mathbb{N} \), is given by
\[
E(W^h) = E \left[ \left( \sum_{j=1}^p W_j \right)^h \right] = (-1)^h \frac{d^h}{dt^h} E(V^t) \bigg|_{t=0} \\
= \sum_{j=1}^N \frac{h!}{\prod_{\ell=1}^p d_{j\ell}!} \prod_{\ell=1}^p \frac{\Gamma(r_{\ell} + d_{j\ell})}{\Gamma(r_{\ell})} \delta_{\ell - d_{j\ell}},
\]
with
\[ N = \sum_{j=1}^{p} \binom{h-1}{j-1} \binom{p}{j} = \binom{h+p-1}{p-1}, \]

and where \( d_{j\ell} \) is a non-negative integer which represents the \( \ell \)-th element in the \( j \)-th partition of \( h \) into \( p \) integers ranging from zero to \( h \), so that
\[ \frac{\Gamma(r_{\ell} + d_{j\ell})}{\Gamma(r_{\ell})} = \frac{d_{j\ell}-1}{\prod_{i=0}^{d_{j\ell}} (r_{\ell} + i)}, \]

with any empty product evaluated as 1.

References