Dividend problems in the dual risk model

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Abstract

We consider the compound Poisson dual risk model, dual to the well known classical risk model for insurance applications, where premiums are regarded as costs and claims are viewed as profits. The surplus can be interpreted as a venture capital like the capital of an economic activity involved in research and development. Like most authors, we consider an upper dividend barrier so that we model the gains of the capital and its return to the capital holders.

By establishing a proper and crucial connection between the two models we show and explain clearly the dividends process dynamics for the former model, properties for different random quantities involved as well as their relations. Using our innovative approach we derive some already known results and go further by finding several new ones. Various authors have addressed the model, we start by making a short overview and then go further. We study different ruin and dividend probabilities, such as the calculation of the probability of a dividend, number of dividends, expected and amount of dividends as well as the time of getting a dividend.

We obtain some integro-differential equations for the above results and also Laplace transforms, then we can get either numerical or analytical results for cases where solutions and/or inversions are possible.

Keywords: Dual risk model; classical risk model; ruin probabilities; dividend probabilities; discounted dividends; dividend amounts; number of dividends.

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1 Introduction

We consider in this manuscript the dual risk model, as explained, for instance, by Avanzi et al. (2007). Using their language, the surplus or equity of a company is explained by the equation,

\[ U(t) = u - ct + S(t), \quad t \geq 0, \tag{1.1} \]

where \( u \) is the initial surplus, \( c \) is a constant meaning the rate of expenses, \( \{S(t), t \geq 0\} \) is a compound Poisson process with parameter \( \lambda \) and density function \( p(x), x > 0 \), of the positive gains, with mean \( p_1 \) (we therefore assume that it exists). Its distribution function is denoted as \( P(x) \). The expected increase per unit time, given by \( \mu = E[S(1)] - c = \lambda p_1 - c \), is positive, that is \( c < \lambda p_1 \). All these quantities have a corresponding meaning in the well known classical continuous time risk model, also known as the Cramér-Lundberg risk model, for insurance applications. For the remainder of our text we will refer this latter model as simply the standard risk model (shortly SM). For those used to work with it we note that the income condition, \( c < \lambda p_1 \), is reversed. A few authors have addressed the dual model (simply DM), we can go back to Gerber (1979) who called it the negative claims model, see pp.: 136-138, also see Bühlmann (1970). We can go even further back to authors like Cramér (1955), Takács (1967) and Seal (1969).

Avanzi et al. (2007), Section 1, explains well where applications of the dual model are said to be appropriate. We just retain a simple but illustrative interpretation, the surplus can be considered as the capital of an economic activity like research and development where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity and gets occasional profits according to a Poisson process. This model has been recently used by Bayraktar and Egami (2008) to model capital investments. Indeed, recently the model has been targeted with several developments, involving the present value of dividend payments and/or dividend strategies. We underline the cited work by Avanzi et al. (2007) and Avanzi (2009), an excellent review paper. Other works are of importance, of which we comment appropriately some lines below.

Important financial applications of the model ruled by (1.1) are the modeling of future dividends of the investments. So, we add an upper barrier, the dividend barrier, noted as \( b(\geq u \geq 0) \). We refer to the upper graph in Figure 1 [see also Figure 1 of Avanzi et al. (2007)]. On the instant the surplus upcrosses de barrier a dividend is immediately paid and the process re-starts from level \( b \). We can also consider the case \( b < u \), however an immediate dividend is paid and the process starts from \( b \), see Avanzi et al. (2007). This makes the situation less interesting from our point of view.

In this manuscript we are not interested on strategies of dividend payments but just the random amounts, once defined their level \( b \). We will consider the payments either discounted or not. Several papers have been published recently using this model with an upper dividend barrier, where the calculation of expected amounts of the discounted paid dividends is targeted. Higher moments have also been considered. See Avanzi et al. (2007), Avanzi and Gerber (2008), Cheung and Drekic (2008), Gerber and Smith (2008), Ng (2009) and Ng (2010). Yang and Zhu (2008) compute bounds for the ruin probability. Song et al. (2008) consider Laplace transforms for the calculation of the expected duration of negative surplus. Cheung (2011) also deals with negative surplus excursions related problems.
For those works as well as in ours where the dividend barrier \( b \) is the key point, it is important to emphasize two aspects: we are going to consider two barriers, one reflecting and another absorbing, the dividend barrier \( b \) and the ruin level “0”, respectively. In the case of the upper barrier \( b \) the process restarts at level \( b \) if this is overtaken by a gain. As mentioned above, this is because an immediate amount of surplus in excess of \( b \) is paid in the form of a dividend, it’s a pay-back capital. It is not the case with the ruin level which makes the process to die down. Indeed, this happens with probability 1 (we’ll come back to this issue later in the text). To achieve a payable dividend the process must not be ruined previously. Furthermore, under the conditions stated the process, sooner or later, will reach one of the two barriers, we mean, with probability one the process reaches a barrier.

In this paper we focus on the connection between the SM and the DM, and based on this we work on unknown problems, however having present some known results from a different viewpoint, which in some cases have interesting interpretations. We will underline these points appropriately. We base our research on the insights and ideas known from the classical risk model. This is a key point for our research. We first do a brief survey of the known results from the literature, then we make important connections between the classical and the dual model features. Afterwards, we make our own developments. We consider important that known results can be looked from our point of view so that our further developments are better taken and understood.

Let’s now consider some of the basic definitions and notation for the dual risk model, those which we address throughout this paper. Some specific quantities we will define and denote on the appropriate section only. First, consider the process as driven by Equation (1.1), free of the dividend barrier. Let

\[ \tau_x = \inf \{ t > 0 : U(t) = 0 | U(0) = x \} , \]

be the time to ruin, this is the usual definition for the model free of the dividend barrier (\( \tau_x = \infty \) if \( U(t) \geq 0 \ \forall t \geq 0 \)). Let

\[ \psi(x, \delta) = E \left[ e^{-\delta \tau} I(\tau_x < \infty) | U(0) = x \right] , \]

where \( \delta \) is a non negative constant. \( \psi(u, \delta) \) is the Laplace transform of time to ruin \( \tau_x \). If \( \delta = 0 \) it reduces to the probability of ultimate ruin of the process free of the dividend barrier, when \( \delta > 0 \) we can see \( \psi(u, \delta) \) as the present value of a contingent claim of 1 payable at \( \tau_x \), evaluated under a given valuation force of interest \( \delta \) [see Ng (2010)].

Let’s now consider an upper level \( b \geq u \geq 0 \) in the model, see the upper graph of Figure 1, we don’t call it yet a dividend barrier. Let

\[ T_x = \inf \{ t > 0 : U(t) > b | U(0) = x \} \]

be the time to reach an upper level \( b \geq x \geq 0 \) for the process which we allow to continue even if it crosses the ruin level “0”. Due to the income condition \( T_x \) is a proper random variable since the probability of crossing \( b \) is one. Dividend will only be due if \( T_x < \tau_x \) and ruin will only occur prior to that upcross otherwise. Whenever we refer to conditional random variables, or distributions, we will denote them by adding a “tilde”, like \( \tilde{T}_x \) for \( T_x | T_x < \tau_x \).

Let’s now introduce into the model the barriers \( b \), as a dividend barrier, and the ruin barrier “0”, respectively reflecting and absorbing, such that if the process isn’t ruined it will
reach the level \( b \). Here, an immediate dividend is paid by an amount in excess of \( b \), the surplus is restored to level \( b \) and the process resumes. We will be mostly working the case \( 0 < u \leq b \).

Let denote the probability of reaching \( b \) before ruin occurring, for a process with initial surplus \( u \), and \( \xi(u, b) = 1 - \chi(u, b) \). We have \( \chi(u, b) = \Pr(T_u < \tau_u) \).

Because of the existence of the barrier \( b \) ultimate ruin has probability 1. The ruin level can be attained before or after the process is reflected on \( b \). Then the probability of ultimate ruin is \( \chi(u, b) + \xi(u, b) = 1 \).

Let \( D_u = \{ U(T_u) - b \text{ and } T_u < \tau_u \} \) be the dividend amount and its distribution function be denoted as

\[
G(u; b; x) = \Pr(T_u < \tau_u \text{ and } U(T_u) \leq b + x | u, b)
\]

with density \( g(u, b; x) = \frac{d}{dx} G(u, b; x) \). \( G(u, b; x) \) is a defective distribution function, clearly \( G(u, b; \infty) = \chi(u, b) \).

We refer now to the upper graph in Figure 1. If the process crosses \( b \) for the first time before ruin at a random instant, say \( T_{(1)} \), then a random amount, denoted as \( D_{(1)} \) is paid. The process repeats, now from level \( b \). The random variables \( D_{(i)} \) and \( T_{(i)} \), \( i = 1, 2, \ldots \), respectively dividend amount \( i \) and waiting time until that dividend, make a bivariate sequence of independent random variables \( \{(T_{(i)}, D_{(i)})\}_{i=1}^{\infty} \). We mean, \( D_{(i)} \) and \( T_{(i)} \) are dependent in general but \( D_{(i)} \) and \( T_{(j)} \), \( i \neq j \), are independent. This follows from the Poisson process properties. This is well known from the classical Cramér-Ludberg risk model. Furthermore, if we take the subset \( \{(T_{(i)}, D_{(i)})\}_{i=2}^{\infty} \) we have now a sequence of independent and jointly identically distributed random variables (and independent of the \( (T_{(1)}, D_{(1)}) \), the bivariate random variables only have the same joint distribution if \( u = b \)). To simplify notations we set that \( (T_{(i)}, D_{(i)}) \) is distributed as \( (T_b, D_b) \), \( i = 2, 3, \ldots \), and \( (T_{(1)}, D_{(1)}) \) is distributed as \( (T_u, D_u) \).

Let \( M \) denote the number of eventual dividends of the process. Total amount of discounted dividends at a force of interest \( \delta > 0 \) is denoted as \( D(u, b, \delta) \) and \( D(u, b) = D(u, b, 0^+) \) is the undiscounted total amount. Their \( n \)-th moments are denoted as \( V_n(u; b, \delta) \) and \( V_n(u; b, \delta) \), respectively. For simplicity denote as \( V(u; b, \delta) = V_1(u; b, \delta) \).

The purpose of this work is to find new results for the different quantities of interest around the dividend problem in the dual risk model as well as to give a new insight for already known results. A key in our work is the interface we establish between the SM and the DM. Despite the reversed income condition, many quantities can be characterized through features well known from the literature regarding the SM. For instance, the single dividend amount random variable can be viewed as the severity of ruin in the SM, although adapted to allow a second barrier.

Our contribution in this work can be split into three different levels. The first contribution is the new insight we give by connecting the DM and the SM. Although this is basically a tool for later developments it is of interest. This is done in Section 3. On a second stage, we develop a new insight and new formulae for an already known problem: expectations for the discounted future dividends. The way we do allow us to find separate formulae for the different stages of dividends, unlike the known formulae from the literature where they appear as an aggregate. This is done in Section 3. On a 3rd level we develop expressions for new quantities such as probability of getting a dividend, number of dividends, amount of a single dividend. The first two quantities are studied in Section 5 whereas Section 6 deals
with the single dividend amount. All of this is done by connecting both the above models. Before, in Section 2 we make an overview of the literature and results for the model. The last Section is devoted to work illustrative examples.

2 Paper review and results

We present here known results from the literature particularly those related to our developments. We are interested on working on the different random variables defined in the previous section and expectations on dividends. It is not our concern dividend strategies, so we omit findings related. Using a martingale argument Gerber (1979) showed that the ruin probability is given by

$$\psi(u) := \psi(u, 0) = e^{-Ru},$$

where $R$ is the unique positive root of the equation

$$\lambda \left( \int_0^\infty e^{-Rx} p(x) dx - 1 \right) + cR = 0. \tag{2.2}$$

We can use a standard probabilistic argument instead, we show it here as the method is going to be used later in the text for other purposes.

If there are no gains until $t_0 = u/c$ ruin level is crossed. By considering whether or not a gain occurs before time $t_0$, we have

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^\infty p(x) \psi(u - ct + x) dx dt,$$

making $s = u - ct$ and rearranging we get

$$ce^{\frac{s}{u}} \psi(u) = c + \int_0^u \lambda e^{\frac{x}{u}} \int_0^\infty p(x) \psi(s + x) dx ds.$$

Differentiating with respect to $u$ we get

$$ce^{\frac{s}{u}} \psi(u) \frac{\lambda}{c} + c e^{\frac{s}{u}} \frac{d}{du} \psi(u) = \lambda e^{\frac{s}{u}} \int_0^\infty p(x) \psi(u + x) dx,$$

from which we get the following integro-differential equation

$$\lambda \psi(u) + c \frac{d}{du} \psi(u) = \lambda \int_0^\infty p(x) \psi(u + x) dx, \tag{2.3}$$

with the boundary conditions $\psi(0) = 1$ and $\psi(\infty) = 0$. Now, it’s easy to set that $\psi(u + x) = \psi(u) \psi(x)$ since ruin to occur from initial level $u + x$ must first cross level $u$ and from there get ruined (due to the independent increments property of the Poisson process). Hence,

$$c \frac{d}{du} \psi(u) = \lambda \psi(u) \left( \int_0^\infty p(x) \psi(x) dx - 1 \right),$$

$$\frac{d}{du} \log \psi(u) = \frac{\lambda}{c} (A - 1) \psi(u),$$

$$\psi(u) = e^{\frac{\lambda}{c} (A-1) u},$$
where \( A = \int_0^\infty p(x)\psi(x)dx \) (\( A \) is not dependent on \( u \)). If we set \( R = -\frac{1}{c}(A-1) \), then we get \( (2.1) \).

Ng (2009) generalized the above probability for a positive \( \delta \), \( \psi(u, \delta) \), which is given by

\[
\psi(u, \delta) = e^{-Ru},
\]

where \( R_\delta \) is the unique positive root such that

\[
\lambda \left( \int_0^\infty e^{-Ru}p(x)dx - 1 \right) + cR_\delta = \delta.
\]

Results below concern expectations, moments, for discounted dividends by integro-differential equations which in some cases shown can be solved analytically. As far as expectations of total dividends is concerned, using a direct and standard approach, by considering possible single gains/jumps events over a small time interval, Avanzi et al. (2007) found that

\[
V(u; b, \delta) = u - b + V(b; b, \delta) \text{ if } u > b, \text{ and }
\]

\[
0 = cV'(u; b, \delta) + (\lambda + \delta)V(u; b, \delta) - \lambda \int_{b-u}^b V(u + y; b, \delta)p(y)dy \quad (2.4)
\]

\[
- \lambda \int_{b-u}^b (u - b + y)p(y)dy - \lambda V(b; b, \delta) [1 - P(b - u)], \text{ if } 0 < u < b,
\]

noting that \( V(0; b, \delta) = 0 \), since ruin is immediate if \( u = 0 \). Besides, Avanzi et al. (2007) found solutions for equation \( (2.4) \) when exponential or mixtures of exponential gains size distributions are considered. Ng (2010) shows solutions when individual gains are phase-type distributed.

For higher moments of discounted dividends, Cheung and Drekic (2008) with a similar procedure show integro-differential equations similar to \( (2.4) \)

\[
V_n(u; b, \delta) = \sum_{j=0}^n \frac{n!}{j!(n-j)!} (u - b)^{n-j} V_j(b; b, \delta) \text{ if } u > b, \text{ and }
\]

\[
0 = cV_n'(u; b, \delta) + (\lambda + n\delta)V_n(u; b, \delta) - \lambda \int_{b-u}^b V_n(u + y; b, \delta)p(y)dy \quad (2.5)
\]

\[
- \lambda \sum_{j=0}^n \frac{n!}{j!(n-j)!} V_j(b; b, \delta) \int_{b-u}^{+\infty} (y - b + u)^{n-j} p(y)dy, \text{ if } 0 < u < b.
\]

as well as solutions for combinations of exponentials distributed gains size and for jump size distributions with rational Laplace transforms. Also, they work an approximation method. We also can get the above results using the approach used for getting \( (2.3) \).

3 Connecting the classical and the dual model

In our further developments it’s crucial the connection between the classical and the dual model as we want to transport methods and results from the first to the second, which has had extensive treatment. So, we put aside together both models, at a first stage we consider
the models free of barriers. As widely known, the standard Cramér-Lundberg risk model is ruled by the equation

$$U^*(t) = u^* + ct - S^*(t), \ t \geq 0,$$  \ (3.1)

where the quantities involved have similar characteristics (although different interpretations) to those corresponding to the dual model. To emphasize that we denote the corresponding quantities with the same letter but coming with a “*”. Apart from their application and interpretation, an important difference between the two models comes with the income condition as noted in the first paragraph of Section 1. We recall that in the SM model that condition comes expressed as $c^* > \lambda^* p_1^*$, which is reversed in our DM model. In the DM this conditions assures that the surplus ultimately tends to infinity with probability one (if no barriers are added). The reversed condition in the DM is intended to achieve the same target, if it wasn’t so the DM would be of difficult application, investors wouldn’t get dividends as they could wish. To compare and relate both models we have to set the DM income condition on both now.

If we first consider first the model without barriers we can relate the two models in the following way

$$U^*(t) = u^* + ct - S(t) = (b - u) + ct - S(t), \ t \geq 0, \ b > u.$$  \ (3.2)

Now, let’s get back to our dividend problem and put the barriers back in order to establish the wanted relation between these two models. We refer to Figure 1. In the DM we consider it with an upper dividend barrier and a ruin barrier. The first is reflecting and the second is absorbing. In the corresponding positive claims model (SM), the corresponding dividend barrier can be seen as the ruin barrier of a surplus starting from initial surplus $b - u$. The other mentioned barrier usually is not considered in the SM, and it may just correspond to an upper line at level $b$. See again Figure 1.

As far as the DM is considered, we note that if the ruin level wasn’t absorbing, i.e., the process would continue if the ruin level “0” was achieved, then the upper level $b$ would be reached with probability 1, due to the income condition. However, we follow the model defined by Avanzi et al. (2007) that we should only pay dividends if the process isn’t ruined. Perhaps we could work with negative capital, but that is out of scope in this work [that kind problem is addressed by Cheung (2011)]. We are only interested working over the set of the sample paths of the surplus process that do not lead to ruin. We need to calculate the probability of the surplus process reaches the barrier $b$ before crossing the level zero. This probability does not correspond to the survival probability, from initial level $u$.

Look at Figure 1, upper graph again. If we turn it upside down (rotate 180°) and look at it from right to left we get the classical model shape, where level “$b$” is the ruin level, “$u$” is the initial surplus, becoming $b - u$, and the level “0” is a upper barrier. $\{D_i\}_{i=2}^{\infty}$ is viewed as a sequence of i.i.d. severity of ruin random variables from initial surplus zero and $D_{(1)}$ the independent, but not identically distributed, severity of ruin random variable from intitial surplus “$b - u$”. Similarly, we have that $\{T_i\}_{i=2}^{\infty}$ can be viewed as a sequence of i.i.d. random variables meaning time of ruin from initial surplus 0, independent of $T_{(1)}$ which in turn represents the time of ruin from initial surplus $b - u$. The connection between the two models is briefly mentioned by Avanzi (2009) (Section 3.1), however not clearly. It is implicit here that in the case of the classical model whenever ruin occurs, the surplus is replaced at level “0”.

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Figure 1: Classical vs dual model
We need to consider some results on the severity of ruin (expectations on the discounted severity of ruin) from the classical risk model adapted to allow an absorbing upper barrier $b > 0$. The following reasoning follows from Dickson and Waters (2004), Section 4, we adapted to consider the barrier $b$ (we refer to Figure 1, $U^*(t)$ graph).

Now, we present some new definitions, valid for this section only. Let $Y_u$ denote the deficit at ruin and $T_u^*$ the time of ruin given an initial surplus $u$. We denote the defective distribution of the deficit $Y_u$ as $G^*(u; b)$ with density $G^*(u; x)$ $[G^*(u; \infty) = \psi^*(u)$, no barrier $b$ considered]. Now, define $\phi_n(u^*, b, \delta) = E[e^{-\delta T_u^*} Y_u^*]$, $n = 0, 1, 2, \ldots$, as an expectation of a discounted power of the severity of ruin ($\phi_1(u^*, b, \delta)$ is the expected discounted severity of ruin). Let $t_0$ denote the time that the surplus takes to reach $b$ if there are no claims, so that $u^* + ct_0 = b$. Using the same approach to set Formula (2.1), by conditioning on the time and the amount of the first claim,

$$
\phi_n(u^*, b, \delta) = \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{u^*+ct} e^{-\delta t} \phi_n(u^* - y, b, \delta) p(y) dy dt
$$

and by substituting $s = u^* + ct$,

$$
\phi_n(u^*, b, \delta) = \frac{1}{c} \int_{u^*}^{b} \lambda e^{-(\lambda+\delta) s - u^*} \int_0^{s} \phi_n(s - y, b, \delta) p(y) dy ds + \frac{1}{c} \int_{u^*}^{b} \lambda e^{-(\lambda+\delta) y - u^*} \int_{s}^{\infty} (y - s)^n p(y) dy ds.
$$

Differentiating and rearranging, we get the following integro-differential equation, with boundary condition $\phi_n(b, b, \delta) = 0$,

$$
\frac{d}{du^*} \phi_n(u^*, b, \delta) = \frac{\lambda + \delta}{c} \phi_n(u^*, b, \delta) - \frac{\lambda}{c} \int_0^{u^*} \phi_n(u^* - y, b, \delta) p(y) dy - \frac{\lambda}{c} \int_{u^*}^{\infty} (y - u^*)^n p(y) dy.
$$

If we denote by $\tilde{\phi}_n(s; b, \delta)$ the Laplace transform of $\phi_n(u^*, b, \delta)$ with respect to $u^*$ after extending the domain of $u$ from $0 \leq u \leq b$ to $u \geq 0$, i.e.,

$$
\tilde{\phi}_n(s; b, \delta) = \int_{0}^{\infty} e^{-su^*} \phi_n(u^*, b, \delta) du
$$

we have

$$
\lambda \tilde{\phi}_n(s; b, \delta) - \lambda \tilde{\phi}_n(s; 0, b, \delta) = (\lambda + \delta) \tilde{\phi}_n(s; b, \delta) - \lambda \tilde{\phi}_n(s; b, \delta) \tilde{p}(s) - \lambda \int_{0}^{\infty} e^{-su^*} \int_{s}^{\infty} (y - u)^n p(y) dy du,
$$

where $\tilde{p}(s)$ is the Laplace transform of the density $p(x)$. The double integral can be re-expressed by using results in Section 1 of Willmot et al. (2005), which are also used in Cheung and Drekic (2008), involving the link between stop-loss moments and equilibrium distributions. So, let $P_n(y)$ be the $n$-th equilibrium function of $P(y)$ (if it exists) such that

$$
P_n(y) = \frac{\int_{0}^{y} \frac{1 - P_{n-1}(t)}{1 - P_{n-1}(t)} dt}{\int_{0}^{\infty} \frac{1 - P_{n-1}(t)}{1 - P_{n-1}(t)} dt}, \quad \text{for } y \geq 0, \ n = 1, 2, \ldots
$$
where \( P_0(y) = P(y) \). The corresponding density and Laplace transform are given by, denoted as \( p_n(y) \) and \( \tilde{p}_n(s) \) respectively,

\[
p_n(y) = \frac{1 - P_{n-1}(y)}{\int_0^1 [1 - P_{n-1}(t)] dt}, \quad \tilde{p}_n(s) = \frac{1 - \tilde{p}_{n-1}(s)}{s \int_0^1 [1 - P_{n-1}(t)] dt}.
\]

Using Equation (1.5) of Willmot et al. (2005), we obtain

\[
\int_u^\infty (y - u)^n p(y) \, dy = p_n(1 - P_n(u)) = p_n \times p_{n+1}(u) \int_0^\infty [1 - P_n(t)] \, dt,
\]

if the \( n \)-th moment \( p_n = \mathbb{E}[X^n] \) exists. Since

\[
\int_0^\infty e^{-su} \int_u^\infty (y - u)^n p(y) \, dy \, du = p_n \tilde{p}_{n+1}(s) \int_0^\infty [1 - P_{n-1}(t)] \, dt = p_n \frac{1 - \tilde{p}_n(s)}{s},
\]

then, from (3.5) we have,

\[
\tilde{\phi}_n(s, b, \delta) = \frac{c\phi_n(0, b, \delta) - \lambda p_n \frac{1 - \tilde{p}_n(s)}{s}}{cs - (\lambda + \delta) + \lambda \tilde{p}(s)}.
\] (3.6)

\( \phi_n(0, b, \delta) \) can be found using the boundary condition above.

These results are going to be used for computation of expectations of times and dividends in the dual model, it will become clear in the next section. Finally, we recall the probability that the surplus attains the level \( b \) without first falling below zero, \( \chi^*(u^*, b) = \delta^*(u^*)/\delta^*(b) \), where \( \delta^*(.) \) is the infinite non-ruin probability. Its complementary comes \( \xi^*(u^*, b) = 1 - \chi^*(u^*, b) \) [please see Dickson (2005), Section 8.2, for details]. We will be using similar probabilities in the dual model.

4 A new approach for the expected discounted dividends

For \( 0 \leq u \leq b \), looking at Figure 1 it’s easy to set the present value of the total dividends in infinite time, or the total discounted dividends amounts, \( D(u, b, \delta) \). Its expected value comes,

\[
V(u; b, \delta) = \mathbb{E}[D(u, b, \delta)] = \mathbb{E} \left[ \sum_{i=1}^{\infty} e^{-\delta \sum_{j=1}^{i} T(j)} D(j) \right] = \sum_{i=1}^{\infty} \mathbb{E} \left[ e^{-\delta \sum_{j=1}^{i} T(j)} D(j) \right]
\]

\[
= \mathbb{E} \left( e^{-\delta T(1)} D(1) \right) + \mathbb{E} \left( e^{-\delta T(1)} \right) \mathbb{E} \left( e^{-\delta T(2)} D(2) \right) + \mathbb{E} \left( e^{-\delta T(1)} \right) \mathbb{E} \left( e^{-\delta T(2)} \right) \mathbb{E} \left( e^{-\delta T(3)} D(3) \right) + \ldots
\]

because the pairs of random variables are independent \( (T(i), D(i)), \ i = 1, 2, \ldots, \) two by two. Note that \( T(i) \) and \( D(i) \) are dependent in general, they have similar properties as time to ruin and its severity in the classical case. Besides, \( (T(i), D(i)), \ i = 2, 3, \ldots, \) are also identically distributed, i.e. \( (T_b, D_b) \overset{d}{=} (T(i), D(i)), \ i = 2, 3, \ldots, \) also \( T(i) \overset{d}{=} T_b, \ i = 2, 3, \ldots \) Set \( (T_n, D_n) \overset{d}{=} (T(1), D(1)), \) we can write

\[
V(u; b, \delta) = \mathbb{E} \left( e^{-\delta T_u} D_u \right) + \mathbb{E} \left( e^{-\delta T_u} \right) \mathbb{E} \left( e^{-\delta T_b} D_b \right) + \mathbb{E} \left( e^{-\delta T_u} \right) \mathbb{E} \left( e^{-\delta T_b} \right) \mathbb{E} \left( e^{-\delta T_b} D_b \right) + \ldots
\]

\[
+ \mathbb{E} \left( e^{-\delta T_u} \right) \mathbb{E} \left( e^{-\delta T_b} \right)^2 \mathbb{E} \left( e^{-\delta T_b} D_b \right) + \ldots
\]

\[
= \mathbb{E} \left( e^{-\delta T_u} D_u \right) + \mathbb{E} \left( e^{-\delta T_u} \right) \mathbb{E} \left( e^{-\delta T_b} D_b \right) \sum_{i=0}^{\infty} \mathbb{E} \left( e^{-\delta T_b} \right)^i
\] (4.1)
Hence
\[ V(u; b, \delta) = \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_b} D_b\right) \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}. \] (4.2)

A similar expression can be found in Dickson and Waters (2004) relating discounted time and severity of ruin in the classical model with a dividend strategy. We only need to evaluate \( \mathbb{E}\left(e^{-\delta T_u} D_u\right), \mathbb{E}\left(e^{-\delta T_b} D_b\right), \mathbb{E}\left(e^{-\delta T_u}\right) \) and \( \mathbb{E}\left(e^{-\delta T_b}\right) \).

To compute the above quantities, and therefore \( V(u; b, \delta) \), we can make use of Expression (3.3) and Equation (3.4) at the end of Section 3, \( \phi_u(u - b, b, \delta) = \mathbb{E}\left(e^{-\delta T_b} D_n^u\right) \) for \( n = 0, 1 \) and \( u^* = 0 \) or \( u - b \). Alternatively, we can invert (3.6).

In the simpler case \( u = b \), we have \( (T_{(1)}, D_{(1)}) \overset{d}{=} (T_b, D_b) \) and \( T_{(1)} \overset{d}{=} T_b \), and the above formula simplifies to
\[ V(b; b, \delta) = \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}. \] (4.3)

Then we have
\[ V(u; b, \delta) = u - b + \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)} \text{ if } u \geq b, \]
because \( V(b; b, \delta) \) is (4.3). Note that Formula (4.2) can be written as
\[ V(u; b, \delta) = \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_b}\right) V(b; b, \delta), \]
which means that the expected value of (discounted) future dividends equals the expected value of the (discounted) first dividend plus the discounted dividends from the process restarting from \( b \).

Unlike Avanzi et al. (2007), \( V(u; b, \delta) \) comes as a function of the expected discounted first dividend amount, of the Laplace transform of the time to get a first dividend and similar quantities for other future dividends and respective times. Then we have to find solutions for all these. For instance, with these formulae we can see the contribution of the first dividend amount (discounted) to the global future dividends. In the last section we can see some numerics.

Using the same method we can compute higher moments. For instance, if we want to compute the variance of the accumulated discounted dividends we need to compute \( V_2(u; b, \delta) \). Let \( Z_i \) be the discounted dividend \( i \) so that
\[ Z_i = e^{-\delta \sum_{j=1}^{i-1} T_{(j)}} D_{(i)}. \]

Then,
\[ V_2(u; b, \delta) = \mathbb{E}[D(u, b, \delta)^2] = \sum_{i=1}^{\infty} \mathbb{E}[Z_i^2] + 2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j]. \]

Using the above properties on the distributions \( (T_b, D_b) \overset{d}{=} (T_{(i)}, D_{(i)}), i = 2, 3, ..., \) and \( (T_u, D_u) \overset{d}{=} (T_{(1)}, D_{(1)}) \) as well as independence in the sequence, we can write
\[ \mathbb{E}[Z_i^2] = \mathbb{E}\left(e^{-2\delta T_u}\right) \mathbb{E}\left(e^{-2\delta T_b}\right)^{i-2} \mathbb{E}\left(e^{-2\delta T_b} D_b^2\right), i = 2, 3, 4, ... \]
\[ \mathbb{E}[Z_i^2] = \mathbb{E}\left(e^{-2\delta T_u} D_u^2\right). \]
Hence,
\[
\sum_{i=1}^{\infty} \mathbb{E}[Z_i^2] = E\left(e^{-\delta T_u D_u^2}\right) + E\left(e^{-\delta T_b D_b^2}\right) \sum_{i=2}^{\infty} \mathbb{E}\left(e^{-\delta T_b}\right)^{i-2} \\
= E\left(e^{-\delta T_u D_u^2}\right) + \frac{E\left(e^{-\delta T_u}\right) E\left(e^{-\delta T_b D_b^2}\right)}{1 - E\left(e^{-\delta T_b}\right)}.
\]

Now,
\[
\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j] = \sum_{j=2}^{\infty} \mathbb{E}[Z_1 Z_j] + \sum_{i=2}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j],
\]
with
\[
\mathbb{E}[Z_1 Z_j] = E\left(e^{-\delta T_u} D_u\right) E\left(e^{-\delta T_b} D_b\right) E\left(e^{-\delta T_b}\right)^{j-2}, j \geq 2 \]
\[
\mathbb{E}[Z_i Z_j] = E\left(e^{-\delta T_u}\right) E\left(e^{-\delta T_u} D_u\right) E\left(e^{-\delta T_b} D_b\right) E\left(e^{-\delta T_b}\right)^{i-2} E\left(e^{-\delta T_b}\right)^{j-i-1},
\]
for \(i < j, i = 2, 3, \ldots\) Then
\[
\sum_{j=2}^{\infty} \mathbb{E}[Z_1 Z_j] = \frac{E\left(e^{-\delta T_u} D_u\right) E\left(e^{-\delta T_b} D_b\right)}{1 - E\left(e^{-\delta T_b}\right)},
\]
\[
\sum_{j=i+1}^{\infty} \mathbb{E}\left(e^{-\delta T_b}\right)^{j-(i+1)} = \sum_{k=0}^{\infty} \mathbb{E}\left(e^{-\delta T_b}\right)^k = \frac{1}{1 - E\left(e^{-\delta T_b}\right)},
\]
\[
\sum_{i=2}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{E}[Z_i Z_j] = \frac{E\left(e^{-\delta T_u}\right) E\left(e^{-\delta T_u} D_u\right) E\left(e^{-\delta T_b} D_b\right) \sum_{i=2}^{\infty} \mathbb{E}\left(e^{-\delta T_b}\right)^{i-2}}{1 - E\left(e^{-\delta T_b}\right)}
\]
\[
= \frac{E\left(e^{-\delta T_u}\right) E\left(e^{-\delta T_b} D_b\right) E\left(e^{-\delta T_b}\right)}{[1 - E\left(e^{-\delta T_b}\right)] [1 - E\left(e^{-\delta T_u}\right)]}.
\]

Function \(V_2(u; b, \delta)\) can be expressed in terms of \(\phi_u(u^*, b, \delta)\) in the following way:
\[
V_2(u; b, \delta) = \phi_2(b - u, b, 2\delta) + \frac{\phi_0(b - u, b, 2\delta)}{1 - \phi_0(0, b, 2\delta)} + \frac{\phi_0(b - u, b, 2\delta)}{1 - \phi_0(0, b, \delta)} + \frac{\phi_1(b - u, b, 2\delta)}{(1 - \phi_0(0, b, \delta))(1 - \phi_0(0, b, 2\delta))}
\]  \hfill (4.4)

We see that we need to evaluate the following six different quantities \(E\left(e^{-\delta T_u}\right), E\left(e^{-\delta T_b}\right), E\left(e^{-\delta T_u} D_u\right), E\left(e^{-\delta T_b} D_b\right), E\left(e^{-\delta T_b} D_b^2\right), E\left(e^{-\delta T_u} D_u^2\right)\), apart from those four needed for the first moment \(E\left(e^{-\delta T_u} D_u\right), E\left(e^{-\delta T_b} D_b\right), E\left(e^{-\delta T_b}\right)\) and \(E\left(e^{-\delta T_u}\right)\).

If we want to consider just the total of a finite number of expected dividends, denoting as \(V(u, b, \delta, n)\), we can compute easily using (4.1)
\[
V(u; b, \delta, n) = \mathbb{E}\left[\sum_{i=1}^{n} e^{-\delta(\sum_{j=1}^{i} T_{(i)})} D_{(i)}\right]
\]
\[
= E\left(e^{-\delta T_u} D_u\right) + E\left(e^{-\delta T_b}\right) E\left(e^{-\delta T_b} D_b\right) \sum_{i=1}^{n-1} \mathbb{E}\left(e^{-\delta T_b}\right)^i
\]
\[
= E\left(e^{-\delta T_u} D_u\right) + E\left(e^{-\delta T_b}\right) E\left(e^{-\delta T_b} D_b\right) \frac{1 - \mathbb{E}\left(e^{-\delta T_b}\right)^n}{1 - \mathbb{E}\left(e^{-\delta T_b}\right)}, n = 2, 3, \ldots (4.5)
\]
and $V(u; b, \delta, 1) = \mathbb{E}(e^{-\delta T_u} D_u)$. The above expression enhances the utility of our approach with new formulae, because it allows to compute the contribution of each dividend to the aggregate expectation as given by Avanzi et al. (2007).

5 On the number of dividends

As we said at the end of Section 1 to get a dividend is necessary that the process reaches/crosses the level $b$ before ruin, which occurs with probability $\chi(u, b)$. The complementary probability is $\xi(u, b)$. Finding closed forms for $\xi(u, b)$, or $\chi(u, b)$, isn’t as straightforward as the similar quantities in the classical model referred at the end of Section 3. Using the usual approach, to reach ruin level prior to dividend level is possible with or without a gain (at time $t_0 : u - ct_0 = 0$). Then, for $0 < u < b$:

$$\xi(u, b) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^b p(x) \xi(u - ct + x, b) dx dt,$$

from which we find

$$\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \int_0^b p(x) \xi(u + x, b) dx$$

or

$$\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \int_u^b p(y - u) \xi(y, b) dy$$

(5.1)

where the boundary conditions is $\xi(0, b) = 1$. Setting $\xi(u + x, b) = \xi(u, b - x) \xi(x, b) = \xi(x, b - u) \xi(u, b)$ we get

$$\lambda \xi(u, b) + c \frac{d}{du} \xi(u, b) = \lambda \xi(u, b) \int_0^{b-u} p(x) \xi(x, b - u) dx$$

$$+ c \frac{d}{du} \xi(u, b) = \xi(u, b) \left( \int_0^{b-u} p(x) \xi(x, b - u) dx - 1 \right)$$

$$\frac{d}{du} \log \xi(u, b) = \frac{\lambda}{c} \left( \int_0^{b-u} p(x) \xi(x, b - u) dx - 1 \right).$$

Likewise, we can get

$$\lambda \chi(u, b) + c \frac{d}{du} \chi(u, b) = \lambda \int_0^{b-u} p(x) \chi(u + x, b) dx + \lambda \left[ 1 - P(b - u) \right]$$

or

$$\lambda \chi(u, b) + c \frac{d}{du} \chi(u, b) = \lambda \int_u^b p(y - u) \chi(y, b) dy + \lambda \left[ 1 - P(b - u) \right]$$

where the boundary conditions is $\chi(0, b) = 0$.

We can compute Laplace transforms on Equation (5.1) as an alternative method to find $\xi(u, b)$. We can use a method of change of variable already used by Avanzi et al. (2007), Section 6, retrieved by Cheung and Drekic (2008) and mentioned in the review paper by Avanzi (2009). In that equation replace $u$ by $z = b - u$ and define $E(z, b) = \xi(b - z, b) = \xi(u, b)$. 

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This change of variable analytically is like setting the relation between the two models, classical and dual. Note that $\mathcal{E}(b, b) = \xi(0, b) = 1$. The corresponding integro-differential equation for $\mathcal{E}(z, b)$ is

$$\lambda \mathcal{E}(z, b) - c \frac{\partial}{\partial z} \mathcal{E}(z, b) - \lambda \int_0^z p(z - y)\mathcal{E}(y, b) \, dy = 0.$$  

In function $\mathcal{E}(z, b)$ extend the range of $z$ from $0 \leq z \leq b$ to $0 \leq z \leq \infty$ and denote the resulting function by $\epsilon(z)$, then compute its Laplace transform, denoted as $\tilde{\epsilon}(s)$, so that

$$\lambda \tilde{\epsilon}(s) - c [s \tilde{\epsilon}(s) - \epsilon(0)] - \lambda \tilde{\epsilon}(s) \tilde{p}(s) = 0.$$  

Hence,

$$\tilde{\epsilon}(s) = \frac{c \epsilon(0)}{cs - \lambda + \lambda \tilde{p}(s)}, \quad (5.2)$$

where $\epsilon(0) = \xi(b, b)$ (note that $\epsilon(b) = \mathcal{E}(b, b) = \xi(0, b) = 1$). When $\tilde{p}(s)$ is a rational function we can invert $\tilde{\epsilon}(s)$ to find a solution for $\epsilon(z)$. Finally $\xi(u, b) = \epsilon(b - u)$ for $0 \leq u \leq b$.

Now let’s consider the multiple dividend situations and let $M$, the number of dividends to be claimed. Then, $\Pr[M = 0] = \xi(u, b)$ and $\Pr[M \geq 1] = 1 - \xi(u, b) = \chi(u, b)$. Besides, $\Pr[M = 1] = \chi(u, b)\xi(b, b)$, first the process crosses $b$ a dividend is paid, then restarts from $b$ and after that is ruined. $\Pr[M = 2] = \chi(u, b)\chi(b, b)\xi(b, b)$, and so on. In summary, we get

$$\begin{align*}
\Pr[M = 0] &= \xi(u, b) \\
\Pr[M = k] &= \chi(u, b)\chi(b, b)^k \xi(b, b), \quad k = 1, 2, ... 
\end{align*}$$

$M$ follows a zero-modified geometric distribution (if $u = b$ we get a geometric distribution with $\Pr[M = k] = \chi(b, b)^k \xi(b, b)$, $k = 0, 1, 2, ...$). The total amount of dividend gains (not discounted, $D(u, b)$) follows a compound zero-delayed geometric distribution.

### 6 On the dividend amount distribution

Now we are going to work on the distribution of the random variable $D_u$, distribution and density functions denoted as $G(u, b; x)$ and $g(u, b; x)$, respectively (see Section 1). First we will set the probability $\xi(u, b)$ and $g(u, b; x)$. Consider the process free of the barrier $b$, just consider $b \geq u$ as a fixed level. We can write, considering that ruin can occur before or after crossing $b$,

$$\begin{align*}
\psi(u) &= \xi(u, b) + \int_0^\infty g(u, b; x)\psi(b + x) \, dx = \xi(u, b) + \psi(b) \int_0^\infty g(u, b; x)\psi(x) \, dx \\
\xi(u, b) &= e^{-Ru} - e^{-Rb} \tilde{g}(u; R)
\end{align*}$$  

(6.1)

setting $\psi(x) = e^{-Rx}$ and rearranging, where $\tilde{g}(u; R)$ is the Laplace transform of the density $g(u, b; x)$ evaluated at $R$.

Back to the usual model, we can compute an integro-differential equation for $G(u, b; x)$ using the standard procedure. Conditioning on the first claim we get, where $t_0$ is such that $u - ct_0 = 0$,

$$G(u, b; x) = \int_0^{t_0} \lambda e^{-\lambda t} \left[ \int_0^{b-(u-ct)} p(y)G(u - ct + y, b; x) \, dy + \int_0^{b-(u-ct)+x} p(y) \, dy \right] dt.$$
Rearranging and differentiating with respect to \( u \), we obtain the following integro-differential equation

\[
\lambda G(u, b; x) + c \frac{\partial}{\partial u} G(u, b; x) = \lambda \int_u^b p(y - u) G(y, b; x) \, dy + \lambda [P(b - u + x) - P(b - u)], \quad (6.2)
\]

with boundary condition \( G(0, b; x) = 0 \). We get easily

\[
\lambda g(u, b; x) + c \frac{\partial}{\partial u} g(u, b; x) = \lambda \int_u^b p(y) g(u + y, b; x) \, dy + \lambda p(b - u + x). \quad (6.3)
\]

We can compute Laplace transforms for \( G(u, b; x) \) by methods similar to those used for (5.2). Let \( G(z, b; x) = G(b - z, b; x) \). Then \( G(b, b; x) = G(0, b; x) = 0 \). Then, from (6.2) we get

\[
\lambda G(z, b; x) - c \frac{\partial}{\partial z} G(z, b; x) - \lambda \int_z^b p(z - y) G(y, b; x) \, dy - \lambda [P(z + x) - P(z)] = 0
\]

Let \( \rho(z, x) \) be the correspondent function arising from extending the range of \( z \). Taking Laplace transforms we get easily,

\[
\lambda \hat{\rho}(s, x) - c \left[ s \hat{\rho}(s, x) - \rho(0, x) \right] - \lambda \hat{\rho}(s, x) \hat{p}(s) + \lambda \left[ \frac{\hat{p}(s)}{s} - \hat{p}(s, x) \right] = 0,
\]

where

\[
\frac{\hat{p}(s)}{s} = \int_0^\infty e^{-sz} P(z) \, dz, \\
\hat{p}(s, x) = \int_0^\infty e^{-sz} \rho(z, x) \, dz, \\
\hat{p}(s, x) = e^{sx} \int_x^\infty e^{-sy} P(y) \, dy. \quad (6.4)
\]

Hence,

\[
\hat{\rho}(s, x) = \frac{c \rho(0, x)}{cs - \lambda} + \lambda \left[ \frac{\hat{p}(s)}{s} - \hat{p}(s, x) \right] / \left( cs - \lambda + \lambda \hat{p}(s) \right) \quad (6.5)
\]

Likewise, for the density \( g(u, b; x) \) from (6.3), setting \( \gamma(z, x) = g(b - z, b; x) \),

\[
\lambda \gamma(z; x) - c \frac{\partial}{\partial z} \gamma(z; x) - \lambda \int_0^z p(z - y) \gamma(y; x) \, dy - \lambda p(z + x) = 0
\]

from which we get the Laplace transform for \( \gamma(z, x) \)

\[
\hat{\gamma}(s, x) = \frac{c \gamma(0, x) - \lambda \hat{\rho}(s, x)}{cs - \lambda + \lambda \hat{p}(s)}
\]

with

\[
\hat{p}(s, x) = e^{sx} \int_x^\infty e^{-sy} p(y) \, dy.
\]

Note that

\[
\frac{\partial}{\partial x} \hat{\rho}(s, x) = \hat{\gamma}(s, x).
\]
Similarly to the classical model with a probability argument we can write
\[
G(u, b; x) = \int_0^{b-u} g(u, u; y)G(u + y, b; x)\,dy + \int_{b-u}^{b-u+x} g(u, u; y)\,dy
\]
\[
g(u, b; x) = \int_0^{b-u} g(u, u; y)g(u + y, b; x)\,dy + g(u, u; b - u + x)
\]

Let’s now consider the process continuing even if ruin occurs. The process can cross for the first time the upper dividend level before or after having ruined. Then we can write the (proper) distribution of the amount by which the process first upcrosses \(b\), denoted as \(H(u, b; x) = \Pr[U(T_u) \leq b + x]\)
\[
H(u, b; x) = H(u, b; x|T_u < \tau_u)\chi(u, b) + H(u, b; x|T_u > \tau_u)\xi(u, b)
\]
\[
= G(u, b; x) + \xi(u, b)H(0, b; x).
\]
The second equation above simply means that the probability of the amount by which the process first upcrosses \(b\) is less or equal than \(x\), equals the probability of a dividend claim less or equal than \(x\) plus the probability of a similar amount but in that case it cannot be a dividend. This second probability can be computed through the probability of first reaching the level \(0\), \(\xi(u, b)\), times the probability of an upcrossing of level \(b\) by the same amount \((\leq x)\) but restarting from \(0\), \(H(0, b; x)\).

We can compute \(H(u, b; x)\) through expressions known for the distribution of the severity of ruin obtained from the positive claims model (recall that the income condition is reversed, making it a proper distribution function). Then we get (the “∗” refers to the classical case)
\[
G^*(b - u; x) = G(u, b; x) + \xi(u, b)G^*(b; x)
\]
equivalent to
\[
G(u, b; x) = G^*(b - u; x) - \xi(u, b)G^*(b; x).
\] (6.6)

For \(u = b\) we have
\[
H(b, b; x) = G(b, b; x) + \xi(b, b)H(0, b; x) \leftrightarrow G(b, b; x) = G^*(0; x) - \xi(b, b)G^*(b; x).
\]

From here differentiate and take Laplace transforms and evaluate at \(R\) we get
\[
g(b, b; x) = g^*(0; x) - \xi(b, b)g^*(b; x)
\]
\[
\bar{g}(b, b; R) = \frac{1}{R} - \frac{\bar{g}^*(0; R)}{\bar{g}^*(b; R)} - \xi(b, b)\bar{g}(b, R),
\]
then use (6.1) to get
\[
\xi(b, b) = \frac{1 - \bar{g}^*(0; R)}{1 - \bar{g}^*(b; R)}e^{-Rb}.
\]
\(g^*(0; x) = \frac{1}{R_{p_1}}[1 - P(x)]\) is the severity density in the positive claims model whose Laplace transform is
\[
\bar{g}^*(0; R) = \frac{1}{R_{p_1}} \left(1 - \int_0^\infty e^{-Rx}p(x)\,dx\right) = \frac{c}{\lambda p_1}
\]
using (2.2). We still need to compute \(\bar{g}^*(b; R)\), clearly, it is not a trivial calculation since it’s a Laplace transform of the severity of ruin with a positive initial surplus in the positive claims model. If \(p(x)\) is exponential then \(\bar{g}^*(0; R) = \bar{g}^*(u; R)\), but that is the trivial example.
7 Illustrations

7.1 Exponential jumps

We consider the case when gain amounts are exponentially distributed, that is $p(y) = \alpha e^{-\alpha y}$, $y > 0$, with $\alpha > 0$. Using Equation (3.4) and then (4.2) we get

$$
\mathbb{E}(e^{-\delta T_u} D_u^n) = \phi_n (b - u, b, \delta) = \frac{n!}{\alpha^n} \frac{\lambda}{c} \frac{e^{-r_2 u} - e^{-r_1 u}}{(r_1 + \alpha)e^{-r_2 b} - (r_2 + \alpha)e^{-r_1 b}} 
$$

where $r_1 < 0$ and $r_2 > 0$ are solutions of the equation

$$
s^2 + \left( \frac{\alpha - \lambda + \delta}{c} \right) s - \frac{\alpha \delta}{c} = 0. 
$$

Expression above for $V(u; b, \delta)$ is equivalent to the one by Avanzi et al. (2007). Function $V_2(u; b, \delta)$ can be evaluated using (4.4), after some simplification we get

$$
V_2(u, b, \delta) = 2 \frac{c \lambda}{\alpha^2} 
$$

$$
\times \frac{(r_1 + \alpha)e^{r_1 b} - (r_2 + \alpha)e^{r_2 b}((c(s_1 + \alpha) - \lambda)e^{-s_2 b} - (c(s_2 + \alpha) - \lambda)e^{-s_1 b})}{((c(r_1 + \alpha) - \lambda)e^{r_1 b} - (c(r_2 + \alpha) - \lambda)e^{r_2 b})((c(s_1 + \alpha) - \lambda)e^{-s_2 b} - (c(s_2 + \alpha) - \lambda)e^{-s_1 b})}
$$

where $s_1$ and $s_2$ are the roots of the equation

$$
s^2 + \left( \frac{\alpha - \lambda + 2\delta}{c} \right) s - \frac{2\alpha \delta}{c} = 0.
$$

For the computation of $\chi(u, b)$ and $\xi(u, b)$, from (5.1) we get, where $R = \lambda/c - \alpha$,

$$
\chi(u, b) = \frac{\lambda - \lambda e^{-Ru}}{\lambda - \alpha ce^{-Rb}},
$$

$$
\xi(u, b) = \frac{\lambda e^{-Ru} - \alpha ce^{-Rb}}{\lambda - \alpha ce^{-Rb}}.
$$

It is much easier to find $\xi(u, b)$ by using (5.2). We get

$$
\epsilon(u) = \frac{(\alpha + R)e^{Ru} - \alpha}{(\alpha + R)e^{Rb} - \alpha}.
$$

Thus $\xi(u, b) = \epsilon(b - u)$.

To find the solution for the distribution of a single amount of dividend, $G(u; b; x)$, we can use the Laplace transforms dealt in Section 6. After some algebra, we get

$$
\rho(u, x) = (1 - e^{-\alpha x}) \left[ 1 - \frac{\alpha - (\alpha + R)e^{Ru}}{(\alpha + R)e^{Rb} - \alpha} \right].
$$

Finally we get,

$$
G(u, b; x) = \rho(b - u, x) = (1 - e^{-\alpha x}) \frac{\lambda - \lambda e^{-Ru}}{\lambda - \alpha ce^{-Rb}}.
$$
Note that we have, for the conditional d.f., denoted as \( \tilde{G}(u, b; x) \),
\[
\tilde{G}(u, b; x) = \frac{G(u, b; x)}{\chi(u, b)} = 1 - e^{-\alpha x}.
\]
We could get the same using (6.6). There is a correspondence to the classical model. Due to the memoryless property of the exponential distribution, the conditional distribution of the single dividend amount has the same distribution of the single gains amount, conditional on the upcross of level \( b \) prior to ruin.

Consider now the conditional distributions, given that \( T_u < \tau_u \). Note that in view of the above we have that
\[
\mathbb{E}\left(e^{-\delta T_u} D_u^n | T_u < \tau_u\right) = \frac{\phi_n(b - u, b, \delta)}{\chi(u, b)}
\]
\[
= \mathbb{E}(D_u^n | T_u < \tau_u) \frac{\lambda}{c} \frac{e^{-r_2 u} - e^{-r_1 u}}{(r_1 + \alpha)e^{-r_2 b} - (r_2 + \alpha)e^{-r_1 b}} \chi(u, b)^{-1}
\]
\[
= \mathbb{E}(D_u^n | T_u < \tau_u) \mathbb{E}\left(e^{-\delta T_u} | T_u < \tau_u\right)
\]
(7.1)

where
\[
\mathbb{E}\left(e^{-\delta T_u} | T_u < \tau_u\right) = \frac{\lambda}{c} \frac{e^{-r_2 u} - e^{-r_1 u}}{(r_1 + \alpha)e^{-r_2 b} - (r_2 + \alpha)e^{-r_1 b}} \chi(u, b)^{-1}.
\]
(7.2)

Expression (7.1) shows that the conditional variables, given that \( T_u < \tau_u \), of time to dividend and dividend amount are independent, respectively \( T_u \) and \( D_u \). This case is the analogue to the classical risk model, between the conditional severity of ruin and time to ruin, given that ruin occurs [see Gerber (1979), for instance]. Hence, Expression (7.2) gives the Laplace transform of the time to dividend.

To get some understanding about the figures we can expect for some of the above quantities we show some numbers. We set a situation where on average we have one jump per unit time with an average amount of one again. That is \( \lambda = \alpha = 1 \) with \( c = 0.75 \) and \( \delta = 0.02 \). We show in Table 7.1 figures for \( V(u, b, 0.02) \) for different values of \( u \) and \( b \). For instance, an investment of three capital units, three times greater than the average jump with a dividend barrier of six (the double of the investment) is expected to give back a present value of capital return of more than seven units. It is more than the double of the invested capital. But with higher dividend levels the returned capital decreases. For more details please see Avanzi et al. (2007) as they find optimal values for the expectation. Table 7.2 shows discounted expected value of a future dividend payment. Note that for \( u = 1 \) and \( b = 1 \) for instance, we get a discounted return of about 45% of the invested capital on the first dividend. Table 7.3 shows figures for the probability of getting a dividend payment, in all cases for the same parameter and argument values. In the of the last table we add a column for the probability of non-ruin \( \chi(u, \infty) = 1 - \psi(u) \).

### 7.2 Combination of exponentials

We use an illustration example also worked by Avanzi et al. (2007), where
\[
p(x) = 3e^{-3x/2} - 3e^{-3x}, \ x > 0.
\]
We consider the same parameter values like in the previous example. Also, the individual gain sizes have mean one, however this example has quite different features. So, in addition to the quantities \( V(u; b; 0.02) \), \( \mathbb{E}(e^{-\delta T_u} D_a) \) and \( \chi(u, b) \) like are shown for the previous example, we thought of some interest to show the quantities \( \mathbb{E}(e^{-\delta T_u}) \) and \( \mathbb{E}(D_u) \). We further show a graph with different plots for the probability density function of the first dividend amounts \( g(u, 10, x) \), for some given values of the initial surplus (see Figure 2). Table 7.4-7.8 show values for \( V(u; b, 0.02) \), \( \mathbb{E}(e^{-\delta T_u} D_a) \), \( \mathbb{E}(e^{-\delta T_u}) \), \( \mathbb{E}(D_u) \) and \( \chi(u, b) \), respectively. In particular, \( \mathbb{E}(e^{-\delta T_u}) \) means the average discounted amount of a first dividend of a one unit amount, and it gives the idea of how much a dividend of one unit is discounted, on average. \( \mathbb{E}(D_u) \) is the undiscounted average dividend. For this example, when compared to the previous one, respective values for the probability of a first dividend \( \chi(u, b) \) are higher.

References


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Table 7.1: Values for $V(u; b, 0.02)$ for $u = 1, 3, 5, 10, 15, 20; b = 2, 3, 6, 10, 30, 40$

<table>
<thead>
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Table 7.2: Values for $\mathbb{E}(e^{-BT_u}D_u)$ for $u = 1, 3, 5, 10, 15, 20; b = 2, 3, 6, 10, 30, 40$

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Table 7.3: Values for $\chi(u, b)$ for $u = 1, 3, 5, 10, 15, 20; b = 2, 3, 6, 10, 30, 40$

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Table 7.4: Values for $V(u; b; 0.02)$ for $u = 1, 3, 5, 10, 15, 20; b = 2, 3, 6, 10, 30, 40$

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Table 7.5: Values for $E(e^{-\delta T_u D_u})$ for $u = 1, 3, 5, 10, 15, 20; b = 2, 3, 6, 10, 30, 40$

<table>
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<tr>
<th>$u$</th>
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Table 7.6: Values for $E(e^{-\delta T_u})$ for $u = 1, 3, 5, 10, 20; b = 2, 3, 6, 10, 30, 40$

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Table 7.7: Values for $E(D_u)$ for $u = 1, 3, 5, 10, 20; b = 2, 3, 6, 10, 30, 40$

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Table 7.8: Values for $\chi(u,b)$ for $u = 1, 3, 5, 10, 20; b = 2, 3, 6, 10, 30, 40$

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</table>
Figure 2: $g(u, 10; x)$, p.d.f of the first dividend.