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ON A CONTINUOUS TIME STOCK PRICE MODEL WITH
REGIME SWITCHING, DELAY, AND THRESHOLD

PEDRO P. MOTA AND MANUEL L. ESQUIVEL

ABSTRACT. Motivated by the need to describe bear-bull market regime switching in stock prices, we introduce and study a stochastic process in continuous time with two regimes, threshold and delay, given by a stochastic differential equation. When the difference between the regimes is simply given by different set of real valued parameters for the drift and diffusion coefficients, changes between regimes depending only on these parameters, we show that if the delay is known there are consistent estimators for the threshold as long we know how to classify a given observation of the process as belonging to one of the two regimes. When the drift and diffusion coefficients are of geometric Brownian motion type we obtain a model with parameters that can be estimated in a satisfactory way, a model that allows to differentiate regimes in some of the NYSE 21 stocks analyzed and also, that gives very satisfactory results when compared to the usual Black-Scholes model for pricing call options.

1. INTRODUCTION

Large literature documents that, usually, in a well defined bull market, with investors realizing small profits frequently, a lower volatility is to be expected. Also, in a installed bear market, with a substantial number of investors trying to sell a the same time, a higher volatility is to be expected. One of our goals in this work is to propose a stock price model able to recover these regimes.

The study of some nonlinear time series models has received renewed attention namely the threshold models (see [1], [2], [9], [10]). In this class, the most popular is the threshold autoregressive model ($TAR(m,p)$), or $TAR(m,p)$, where the process is divided into $m$ regimes following in each regime an $AR(p)$ model. One of our goals is to study an extension of threshold processes to continuous time and to obtain estimation methods for the parameters of this kind of processes. A diffusion which experiences a regime change when crossing over a threshold $m$, will be our generic model for the stochastic process. In [3], [7] some results for threshold continuous time processes are given, however the changes in the regimes are a consequence of the process hitting an upper threshold when the process was in the first regime, or an lower threshold when the process was in the second regime. Other regime switching diffusion models are studied, for instance in [4], with the regime changes being driven by an independent Poisson process.

In this paper we consider stochastic processes defined by a stochastic differential equation where the drift and (possibly) the volatility coefficients change from

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one regime to another. In each regime the process follows the dynamics of a simple continuous time process and the change in regime will happen when the process crosses a threshold; this motivates the denomination continuous time delayed threshold regime switching process (DTRS). At each time $t$ the process is in the first regime, if at time $t - d$, where $d$ is the delay parameter, the process is below the threshold $m$ or in the second regime if is above the threshold. We note that the regime switching here proposed is not driven by some external random source; it only depends on the possible crossings of thresholds by the trajectories of the process.

Our overall goal is to define and study some main aspects of the model, namely, parameter estimation and usefulness for practical applications. Given a set of vector parameters $\theta \in \Theta \subset \mathbb{R}^r$ the process under study could be given in full generality by

$$
\begin{align*}
&dX_t = \mu(t, X_t, X_{t-d}, \theta)dt + \sigma(t, X_t, X_{t-d}, \theta)dB_t \\
&X_0 = x_0,
\end{align*}
$$

(1.1)

where, being interested in a two regime switching model with a threshold parameter $m$ and delay parameter $d$, we have $\Theta = \{(\mu_1, \sigma_1), (\mu_2, \sigma_2)\}$ and also:

$$
\begin{align*}
&\mu(t, X_t, X_{t-d}, \theta) = \mu(t, X_t, \mu_1)1_{\{X_{t-d} \leq m\}} + \mu(t, X_t, \mu_2)1_{\{X_{t-d} > m\}}, \\
&\sigma(t, X_t, X_{t-d}, \theta) = \sigma(t, X_t, \sigma_1)1_{\{X_{t-d} \leq m\}} + \sigma(t, X_t, \sigma_2)1_{\{X_{t-d} > m\}}.
\end{align*}
$$

We are interested in the study of this process when both $\theta_1 = (\mu_1, \sigma_1)$, and $\theta_2 = (\mu_2, \sigma_2)$, are such that the process, in the long run, will keep dynamics with the two regimes. This will happen, for instance, if $\mu_1 > 0$ and $\mu_2 < 0$ or, $\mu_1 < 0$ and $\mu_2 > 0$.

The third section is devoted to the definition of consistent estimators when we have more information than the one given by the observations $X_1, X_2, ..., X_n$ of the process, that is, we will suppose that the delay $d$ is known and the regime to which each observation belong is also known. The fourth section provides a conditional least squares (CLS) estimating procedure, based on the results of the third section, where we will estimate all the process parameters using only the information from the observations $X_1, X_2, ..., X_n$ of the process. In this general setting we do not have, for now, consistency results. In the three last sections we present a simulation study for particular processes, using the CLS estimators approach introduced, an application to real data and a comparison benchmark study between the model proposed and the Black-Scholes model for pricing call options.

2. On the existence of a solution for the model

In a context of practical applications, like pricing contingent claims, there is no generality loss if we suppose that $t \in [0, T]$ with $T$ a deterministic finite time horizon. This hypothesis, in turn, will contribute to simpler proofs. In order to define $(X_t)_{t \in [0, T]}$, a solution to equation (1.1), we will suppose that for $t = 0$ a parameter choice, for instance, $\theta_1 = (\mu_1, \sigma_1)$ will give diffusion coefficients $\mu(t, X_t, X_{t-d}, \theta_1) = \mu(t, X_t, \mu_1)$ and $\sigma(t, X_t, X_{t-d}, \theta_1) = \sigma(t, X_t, \sigma_1)$ such that the
stochastic differential equation
\[
\begin{align*}
\left\{ \begin{array}{l}
\,dX_t^{\theta_1} = \mu(t, X_t^{\theta_1}, \mu_1)dt + \sigma(t, X_t^{\theta_1}, \sigma_1)dB_t \\
X_0^{\theta_1} = x_0,
\end{array} \right.
\end{align*}
\]
has a unique continuous solution \((X_t^{\theta_1,x_0})_{t \in [0,T]}\) for the chosen time horizon \(T > 0\) (see [5, p. 289] or [6, p. 73]). Let now \(\tau_1\) be the first stopping time at which the solution process crosses the threshold \(m\), for instance, from below, that is:
\[
\tau_1 := \inf \left\{ 0 < t < T : \forall \epsilon > 0, \exists t^-_\epsilon, t^+_\epsilon > 0, t - \epsilon < t^-_\epsilon < t < t^+_\epsilon < t + \epsilon, \right. \\
X_{t^-_\epsilon}^{\theta_1,x_0} \leq m \land X_{t^+_\epsilon}^{\theta_1,x_0} > m \left. \right\}
\]
\(X_{\tau_1}^{\theta_1,x_0} = m\). By definition, for \(0 \leq t \leq \tau_1 + d \leq T\), we will have that \(X_t = X_t^{\theta_1,x_0}\). Now, a regime switch having occurred at time \(\tau_1 + d\) consider (if \(\tau_1 + d < T\)) the stochastic differential equation (with a random initial condition depending on a stopping time) given by:
\[
\begin{align*}
\left\{ \begin{array}{l}
\,dX_t^{\theta_2} = \mu(t, X_t^{\theta_2}, \mu_2)dt + \sigma(t, X_t^{\theta_2}, \sigma_2)dB_t, \, \tau_1 + d \land T \leq t \leq T \\
X_{\tau_1+d,T}^{\theta_2} = x_1 := X_{\tau_1+d,T}^{\theta_1,x_0}.
\end{array} \right.
\end{align*}
\]
Note that this equation should be read, for all \(t \in [0, T]\), as
\[
\begin{align*}
X_t^{\theta_2} &= X_{\tau_1+d,T}^{\theta_1,x_0} \mathbb{1}_{[\tau_1+d,T]}(t) + \int_0^t \mu(u, X_u^{\theta_2,x_0}, \mu_2)\mathbb{1}_{[\tau_1+d,T]}(u) \, du + \\
&\qquad + \int_0^t \sigma(u, X_u^{\theta_2,x_0}, \sigma_2)\mathbb{1}_{[\tau_1+d,T]}(u) \, dB_u
\end{align*}
\]
and then it is clear that if the diffusion coefficients and initial condition \(Z\) of the following equation
\[
\begin{align*}
\left\{ \begin{array}{l}
\,dX_t^s = \mu(t, X_t^s, \mu_2)dt + \sigma(t, X_t^s, \sigma_2)dB_t, \, s \leq t \leq T \\
X_0^s = Z
\end{array} \right.
\end{align*}
\]
are such that an unique continuous solution exists for all \(s \in [0, T]\), then a continuous unique solution \((X_t^{\theta_2,x_1})_{t \in [\tau_1+d,T]}\) exists also for equation 2.2. In fact, consider a standard theorem, for instance, the one in [5, p. 289]. Then, as
\[
\mathbb{E}\left[ |X_{\tau_1+d,T}^{\theta_1,x_0} - X_{\tau_1+d,T}^{\theta_2,x_1}(t)|^2 \right] \leq \mathbb{E}\left[ |X_{\tau_1+d,T}^{\theta_1,x_0}|^2 \right] \leq \mathbb{E}\left[ \sup_{0 \leq t \leq T} |X_t^{\theta_1,x_0}|^2 \right] < +\infty,
\]
the sufficient condition for the initial value is verified. Moreover, for all \(t \in [0, T]\) as
\[
|\mu(t, X_t^{\theta_1,x_0}, \mu_2)\mathbb{1}_{[\tau_1+d,T]}(t)| \leq |\mu(t, X_t^{\theta_1,x_0}, \mu_2) |
\]
and
\[
|\sigma(t, X_t^{\theta_1,x_0}, \sigma_2)\mathbb{1}_{[\tau_1+d,T]}(t)| \leq |\sigma(t, X_t^{\theta_1,x_0}, \sigma_2) |
\]
it is clear that the integrability, Lipschitz and sub-linear growth conditions verified by the diffusion coefficients of equation (2.3) are still verified by the diffusion coefficients of equation (2.2).
As before, let now \(\tau_2\) be the first stopping time following \(\tau_1\) at which the process crosses the threshold \(m\), but now from above, that is:
\[
\tau_2 := \inf \left\{ \tau_1 < t < T : \forall \epsilon > 0, \exists t^-_\epsilon, t^+_\epsilon > 0, t - \epsilon < t^-_\epsilon < t < t^+_\epsilon < t + \epsilon, \\
X_{t^-_\epsilon}^{\theta_1,x_0} > m \land X_{t^+_\epsilon}^{\theta_1,x_0} \leq m \right. \left. \right\}
\]
and \(X_{1}^{\theta_{2},x_{1}} = m\). By definition, for \(\tau_{1} + d \leq t \leq \tau_{2} + d \leq T\) we will have that \(X_{t} = X_{1}^{\theta_{2},x_{1}}\). The solution process to equation (1.1) may be defined inductively in this way by gluing together solutions to standard stochastic differential equations that the diffusions cross the threshold \(m\) defined between stopping times. These stopping times are the times at which the observations cross the threshold \(m\) translated by the delay \(d\).

3. Consistent estimators

Under some restrictive hypothesis it is possible to define consistent estimators of the threshold \(m\). Let, for each integer \(n\), be \(C_{n} = \{X_{t_{1}}, X_{t_{2}}, ..., X_{t_{n}}\}\) be the observations of the DTRS process at times \(t_{1}, t_{2}, ..., t_{n}\), not necessarily equally spaced, with \(\Delta_{i}^{n} = t_{i+1} - t_{i}, i = 1, ..., n - 1\) being the decreasing time interval between two consecutive observations such that \(\lim_{n \to +\infty} \max_{1 \leq i \leq n} \Delta_{i}^{n} = 0\). We admit that in the observation protocol, from one step to the next, we keep the observations from the previous step, this implying that for each \(n\), \(C_{n} \subset C_{n+1}\).

We suppose that we observe the random variables \(R_{1}, R_{2}, ..., R_{n}\) where \(R_{i} = 1\) if \(X_{t_{i}}\) belongs to regime 1 or \(R_{i} = 2\) if \(X_{t_{i}}\) belongs to regime 2. We suppose also that the delay parameter \(d\) is known.

With these hypothesis we can define a estimator for the threshold \(m\). In fact, knowing the sequence \(R_{1}, R_{2}, ..., R_{n}\) and the delay \(d\) we can split the the observations \(X_{t_{1}}, X_{t_{2}}, ..., X_{t_{n}}\) into two sets using the fact that: \(R_{j} = 1 \iff X_{t_{j} - d} \leq m\). For that, if \(R_{i} = 1\) define

\[
\tilde{X}_{i}^{-} = \begin{cases} 
X_{t_{i} - d} & \text{if } X_{t_{i} - d} \text{ is an observation} \\
\min(X_{t_{j} - d}, X_{t_{j+1} - d}) & \text{for } t_{i} - d \in [t_{j}, t_{j+1}] \text{ if } X_{t_{i} - d} \text{ is not an observation.}
\end{cases}
\]

Then, the set

\[
C_{n}^{-} = \{\tilde{X}_{i}^{-} : R_{i} = 1, i = 1, ..., n\}
\]

is the set of observations that are smaller or equal than \(m\), and similarly if \(R_{i} = 2\) for

\[
\tilde{X}_{i}^{+} = \begin{cases} 
X_{t_{i} - d} & \text{if } X_{t_{i} - d} \text{ is an observation} \\
\max(X_{t_{j} - d}, X_{t_{j+1} - d}) & \text{for } t_{i} - d \in [t_{j}, t_{j+1}] \text{ if } X_{t_{i} - d} \text{ is not an observation,}
\end{cases}
\]

the set

\[
C_{n}^{+} = \{\tilde{X}_{i}^{+} : R_{i} = 2, i = d, ..., n\}
\]

is such that the observations are strictly larger than \(m\).

Finally, we estimate the threshold \(m\) in a consistent way.

**Theorem 3.1.** If the process \((X_{t})_{t \geq 0}\) has continuous trajectories and if there is at least a change from the first to the second regime then with

\[
\hat{m}_{n}^{-} = \max C_{n}^{-} \text{ we have } \lim_{n \to +\infty} \hat{m}_{n}^{-} = m \text{ a.s.}
\]

that is, \(\hat{m}_{n}^{-}\) is a consistent estimator of the threshold \(m\). And if there is at least a change from the second to the second regime then with

\[
\hat{m}_{n}^{+} = \min C_{n}^{+} \text{ we have } \lim_{n \to +\infty} \hat{m}_{n}^{+} = m \text{ a.s.}
\]

that is, \(\hat{m}_{n}^{+}\) is a consistent estimator of the threshold \(m\).
Proof. We will prove that $\hat{m}_n^- = \max C_n^-$ is a strongly consistent estimator. Observe that as $C_n^- \subset C_{n+1}^-$ we have that $\hat{m}_n^- \leq \hat{m}_{n+1}^-$ and that for each $n$, by definition, $\hat{m}_n^- \leq m$. This implies that $\lim_{n \to \infty} \hat{m}_n^- = m$ (and that $\limsup_{n \to +\infty} \hat{m}_n^- \leq m$).

Suppose that

$$\lim_{n \to +\infty} \hat{m}_n^- = \lim_{n \to +\infty} \left( \max C_n^- \right) = \limsup_{n \to +\infty} \left( \max C_n^- \right) < m .$$

Then, for some $\varepsilon > 0$, there exists a $p \geq 1$ such that for all $n \geq p$:

$$(3.5) \quad \hat{X}_i^- \in C_n^- \Rightarrow \hat{X}_i^- < m - \varepsilon.$$  

Let $\tau$ be the first random time at which the process has a change in regime from the first to the second. Recall that this implies that $X_{\tau-d} = m$. As the process has continuous trajectories, for $t \leq \tau - d$ we have that $X_t \leq m$ and, for $t > \tau - d$ but close enough to $\tau - d$ (before another regime change), we have $X_t > m$.

Now choose $\delta = \delta(\omega)$ such that for all $t$ verifying $| (\tau - d) - t | < \delta$ we have $| X_t(\omega) - X_{\tau-d}(\omega) | < \varepsilon$ and choose $q = q(\varepsilon, \omega)$ such that for $n \geq q$ we have that $\max_{1 \leq i \leq n} \Delta_n^i < \delta$.

Then, for $n \geq \max(q,p)$ and an observation time $t_i = t_i(\omega) \in \{1, \ldots, n\}$ such that $\tau - d \in [t_i, t_{i+1}]$ (which exists due to the hypothesis on the observations), that is such that $X_{t_i} \in C_n^-$ (as $t_i < \tau - d$), we have that

$$| X_{t_i}(\omega) - X_{\tau-d}(\omega) | = | X_{t_i}(\omega) - m | < \varepsilon ,$$

a contradiction with (3.5).

In the same way we can conclude that the estimator $\hat{m} = \min C_n^+$ is also consistent and, of course, any estimator of the form $\hat{m}_n(\lambda) = \lambda \max C_n^- + (1 - \lambda) \min C_n^+$ for $\lambda \in [0,1]$ in case there at least one regime change from the first to the second and one the other way around.

Remark 3.2. An open problem is to define a consistent estimator for the threshold without the restrictions above, namely, the known classification of the observations in the two regimes and the assumption that the delay is known. One way to relax the restrictions could be to define a consistent estimator for the delay. Overcoming the need for the regime classification seems to be a hard problem.

4. Conditional least squares estimators

In this section our purpose is to implement a practical way of getting an estimator for the delay, the threshold and for the other parameters of the process when we do not know the regime for each observation. The first step of the procedure consists in the classification of the observations in regimes in order to built an conditional least squares function. For each $n$ the procedure is implemented as follows.

1. For fixed delay and threshold, respectively $d$ and $m$, we split the observations in two regimes, $\hat{R}_1(d,m)$ and $\hat{R}_2(d,m)$ corresponding to the regime with positive trend and negative trend respectively. Naturally we limit our search for $d$ taking the values $p\Delta, p = 1, \ldots, n - 1$ and $m \in \{X_{1:n}, \ldots, X_{n:n}\}$ but in such a way that at least each regime have 15% of the total observations. The regime classification is done in the following way: we start with the observation $X_1$ and if $X_1 \leq m$ we consider $X_{1+d}$ in the first regime
(\hat{R}_{1+d} = 1) or else we consider \(X_{1+d}\) in the second regime (\(\hat{R}_{1+d} = 2\)).

Next, we look at \(X_2\) and repeat the classification procedure for the observation \(X_{2+d}\), continuing until the end of the observations.

(2) Next, we can compute the conditional estimators for the diffusion parameters, namely \(\mu(t, X_t, \theta)\) the drift, and \(\sigma(t, X_t, \theta)\) the volatility, using the observations in each of the regimes and the usual estimators for the parameters in the simple process context.

(3) We define the conditional least squares function,

\[
CLS_n(d, m) = \sum_{j \geq d} (X_{j+1} - E_{\mu, \sigma, d, m}[X_{j+1}|X_j, ..., X_1])^2
\]

(4) Finally we choose as delay and threshold estimators the values that minimize \(CLS_n(d, m)\), that is,

\[
(d_n, m_n) = \text{argmin}_{(d, m)} CLS_n(d, m).
\]

Remark 4.1. The conditional expectation in equation (4.1) is not explicitly known and we will approximate it by the conditional expectation of the underlying process, in each regime.

We conjecture that this procedure will give consistent estimators when \(\Delta\) decreases and simultaneously the interval where we observe the process, \([0, t_n]\), increases. To get asymptotical results we should need a decreasing \(\Delta\) because that is the only way to ensure that the regime classification of the observations converges to the true one for some sequence of delays and thresholds \((d_n, m_n)_{n \in \mathbb{N}}\). At the same time, we need to increase the observation interval \([0, t_n]\) to ensure the consistency of the other process parameter estimators.

4.1. Simulation study. We are now in conditions to illustrate this procedure with a simple simulation study for a concrete process. We will define the geometric Brownian motion with threshold (GBMT) as,

\[
dX_t = \mu(t, X_t, X_{t-d}, \theta)dt + \sigma(t, X_t, X_{t-d}, \theta)dB_t, \quad X_0 = x_0,
\]

where \(\theta \in \Theta = \{ (\mu_1, \sigma_1), (\mu_2, \sigma_2) \} \subset \mathbb{R}^2\)

\[
\mu(t, X_t, X_{t-d}, \theta) = \mu_1 X_t \mathbb{I}_{[X_{t-d} \leq m]} + \mu_2 X_t \mathbb{I}_{[X_{t-d} > m]}, \quad \mu_1 > 0, \mu_2 < 0,
\]

\[
\sigma(t, X_t, X_{t-d}, \theta) = \sigma_1 X_t \mathbb{I}_{[X_{t-d} \leq m]} + \sigma_2 X_t \mathbb{I}_{[X_{t-d} > m]}, \quad \sigma_1, \sigma_2 > 0.
\]

The trajectory of the process is generated using a discretization step of .01 and we start the procedure with the conditional regime classification. The following figure illustrates a part of the trajectory under consideration.

The simulations were done using the Mathematica\textsuperscript{TM} software and firstly we start with 100 repetitions with 1000 observations in each repetition, considering the process parameters, \(\mu_1 = 1, \mu_2 = -1, d = 30, m = 10\), with different values for \(\sigma\).

For the estimating procedure we introduce the auxiliary conditional least squares contrast function,

\[
CLS_n(d, m) = \sum_{k=1}^{2} \sum_{i \geq d} \left( X_{i+1} - X_i e^{-\hat{\mu}_k(d, m) \Delta} \right)^2 1_{\hat{R}_i(d, m) = k},
\]
We perform a grid search for $d$ in $\{1, 60\}$ and for $m$ in $[7, 13]$ with grid step of 0.05.

The results for the different values of $\sigma$ under consideration are presented in the next table.

**Table 1.** Estimates for the GBMT process with $\mu_1 = 1$, $\mu_2 = -1$, $d = 30$ and $m = 10$, and for different values of $\sigma$

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>mean</th>
<th>$d$</th>
<th>$m$</th>
<th>$\hat{\mu}_1$</th>
<th>$\hat{\mu}_2$</th>
<th>$\hat{\sigma}_1$</th>
<th>$\hat{\sigma}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>30</td>
<td>10</td>
<td>0.9836</td>
<td>-1.0105</td>
<td>0.9908</td>
<td>0.1011</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>30</td>
<td>10</td>
<td>0.9334</td>
<td>-1.0846</td>
<td>0.9548</td>
<td>0.0974</td>
<td>0.1053</td>
</tr>
<tr>
<td>0.5</td>
<td>31.8700</td>
<td>10</td>
<td>0.8606</td>
<td>-1.1603</td>
<td>0.4957</td>
<td>0.5061</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>30</td>
<td>10</td>
<td>0.7692</td>
<td>-1.3329</td>
<td>0.6948</td>
<td>0.7128</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>34.8489</td>
<td>10</td>
<td>0.4224</td>
<td>-1.7332</td>
<td>1.0029</td>
<td>1.0196</td>
<td></td>
</tr>
</tbody>
</table>

As we can see, the results suggest that the procedure works well, getting good approximations for the original values and, as expected, the standard deviation for the estimators are larger when $\sigma$ increases.

5. **Real data**

Now we will apply the procedure to real data gathered from Yahoo Finance. The data consists of stock daily prices of 21 companies. In the following table we present the results of the estimation procedure. It is to be remarked that for most of the stocks the delay $d$ is equal to one day, the first drift $\hat{\mu}_1$ is positive, the second drift $\hat{\mu}_2$ is negative while the first volatility $\hat{\sigma}_1$ is larger than the second $\hat{\sigma}_2$. This observation is not consistent with more volatility being associated with a decreasing trend than with an increasing one. This results from having two regimes with different drifts and volatilities; in the geometric brownian motion model, the volatility is a measure of the deviation of the logarithmic prices towards the mean. In the proposed two regime model with a positive and a negative drift it is possible to have large down price movements with small deviations to the implicit mean, thus with smaller volatility. Noticeable exceptions are Google, Apple, Starbucks and Motorola with delays 13, 8, 15 and 14 respectively. Other noticeable exceptions are the New York Times with both drifts negative, Microsoft with both drifts positive.
and Google, McDonalds, Monsanto and Amazon with the first volatility smaller than the second one.

### Table 2. Estimated parameters for various stocks; data range from January 2005 to November 2009

<table>
<thead>
<tr>
<th>Stock</th>
<th>d</th>
<th>m</th>
<th>μ₁</th>
<th>μ₂</th>
<th>σ₁</th>
<th>σ₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>Google</td>
<td>13</td>
<td>642</td>
<td>-0.00160896</td>
<td>-0.00589534</td>
<td>0.0232655</td>
<td>0.0252965</td>
</tr>
<tr>
<td>HP</td>
<td>1</td>
<td>46.9</td>
<td>-0.0010484</td>
<td>-0.0031286</td>
<td>0.0246834</td>
<td>0.0171097</td>
</tr>
<tr>
<td>Apple</td>
<td>8</td>
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<td>-0.0044622</td>
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<td>-0.00294712</td>
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<td>-0.00921267</td>
<td>0.0333572</td>
<td>0.0154744</td>
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<td>Cisco</td>
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<td>0.0124365</td>
<td>-4.15510</td>
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<td>-0.0037116</td>
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<td>JP Morgan</td>
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<td>-0.00911782</td>
<td>0.0827172</td>
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<td>McDonalds</td>
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<td>-0.00117256</td>
<td>0.0149628</td>
<td>0.0162239</td>
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<td>Starbucks</td>
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<td>0.005457814</td>
<td>0.00157847</td>
<td>0.028605</td>
<td>0.0164251</td>
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<td>Philip Morris</td>
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<td>0.00481041</td>
<td>-0.00169004</td>
<td>0.0297844</td>
<td>0.0201584</td>
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<tr>
<td>P &amp; G</td>
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<td>0.0076696</td>
<td>-0.0010604</td>
<td>0.0128283</td>
<td>0.0126707</td>
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<tr>
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<td>-0.00076694</td>
<td>0.0222837</td>
<td>0.0159927</td>
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<tr>
<td>PG&amp;E</td>
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<td>-0.00016351</td>
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<td>0.0130605</td>
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<td>-0.00053459</td>
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</tr>
<tr>
<td>Motorola</td>
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<tr>
<td>Monsanto</td>
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<td>Amazon</td>
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<td>0.0017529</td>
<td>-0.0013420</td>
<td>0.0342596</td>
<td>0.0209723</td>
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</table>

In figure 2 we present a graphical representation of the estimated regimes and thresholds for these stock prices. It is visually clear that the existence of regimes and a threshold has plain sense for some of the stocks, for instance P & G, CA, Adobe, HP while for Coca Cola, JP Morgan and PG&E the second regime appears as residual and so the model doesn’t seem to be justified. A ratio statistical test on the difference of the two regime volatilities showed that, with a confidence level of 99%, only for P & G, Google Apple, IBM and Monsanto we reject the hypothesis of these volatilities being different. A determination of the confidence intervals for the two regime volatilities of Adobe, CA, HP and P & G, shows that only for P & G do these confidence intervals overlap.

### 6. Pricing

We will consider a market with two securities as in the generalized Black-Scholes model of [8, p. 288]. We suppose that we are working with a constant interest rate equal to $r$. With the notations of (1.1) and (4.4) and knowing that $X_t \neq 0$ we have that the Novikov condition is satisfied,

$$
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \left( \frac{\mu(t, X_t) - r}{\sigma(t, X_t)} \right)^2 dt \right) \right] = \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 dt \right) \right] =
$$

$$
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \left( \frac{\mu_1 - r}{\sigma_1} \right)^2 dt \right) \mathbb{I}_A + \exp \left( \frac{1}{2} \int_0^T \left( \frac{\mu_2 - r}{\sigma_2} \right)^2 dt \right) \mathbb{I}_{A^c} \right] < +\infty
$$

with $A = \{ \omega \in \Omega : \mu_1(\omega) = \mu_1, \sigma_1(\omega) = \sigma_1 \}$. Being so, the market is arbitrage free and complete, the price of a contingent $T$-claim $f(X_T)$, given non random price $X_{t_0}$, is unique and it is given by the usual $e^{-r(T-t_0)}\mathbb{E}[f(X_T)]$, with $\mathbb{Q}$ the martingale measure.

We now detail some simulation implementation remarks relevant for price calculation. In the simulation study we use the Euler scheme to generate the trajectories...
of the process. If we have a geometric Brownian motion process without regimes the Euler scheme for a discretization \( t_0, \ldots, t_n \) with constant step size \( \Delta = t_{n+1} - t_n \), is:

\[
X_{i+1} = X_i + \mu(t_i, X_i)\Delta + \sigma(t_i, X_i)\Delta B_i
\]

with \( \mu(t_i, X_i) = \mu X_i, \sigma(t_i, X_i) = \sigma X_i \) and \( \Delta B_i \), a step \( \Delta \), increment of standard Brownian motion.

The natural extension to use, for the process with regimes is the following approximation:

\[
X_{i+1} = X_i + \mu_i(t_i, X_i)\Delta + \sigma_i(t_i, X_i)\Delta B_i
\]

where

\[
\mu_i(t_i, X_i) = \mu_1 X_i^{\mathcal{I}_{\{X_{i-\Delta} \leq m\}}} + \mu_2 X_i^{\mathcal{I}_{\{X_{i-\Delta} > m\}}}
\]

and

\[
\sigma_i(t_i, X_i) = \sigma_1 X_i^{\mathcal{I}_{\{X_{i-\Delta} \leq m\}}} + \sigma_2 X_i^{\mathcal{I}_{\{X_{i-\Delta} > m\}}}
\]

and \( d \) the delay. We must stress that we need to choose a discretization step such that, \( \Delta = \frac{d}{k} \) for some integer \( k \), valid only for \( t_i \geq d \). To account for the initial part of the trajectories in the time interval \( [0, d] \), we generate the observations using the Euler approximation for the geometric Brownian motion without regimes (choosing one of the regimes) and after that we can generate the observations of the process with regimes, using formula (6.1).

A validation of the procedure just described was done observing that by non-arbitrage and for a non random \( X_t \), \( \mathbb{E}^Q[X_T] = X_t e^{r(T-t)} \). In the Euler approximation context we also have a similar asymptotic relation with \( X_t \) random,

\[
\mathbb{E}^Q[X_T] = \mathbb{E}[X_T]^Q \approx \mathbb{E}[X_t] e^{r(T-t)}
\]

when \( n \), the number of discretization steps, goes to infinity. In fact, the following is true.

**Lemma 6.1.** With \( X_n^Q = X_T^Q = X_t \prod_{i=1}^n (1 + r\Delta + \sigma_i \Delta B_i) \) we have that:

\[
\mathbb{E}^Q[X_T] = \mathbb{E}[X_t] e^{r(T-t)}.
\]

**Proof.** As \( X_{k-1}^Q \) and \( \sigma_k \) are measurable with respect to the \( \sigma \)-algebra given by \( \mathcal{F}_{k-1} = \mathcal{F}(X_0, \ldots, X_{k-1}) \) and \( \Delta B_k \) is independent of \( \mathcal{F}_{k-1} \) with \( \mathbb{E}[\Delta B_k] = 0, k = 1, \ldots, n \) we may write:

\[
\mathbb{E}[X_n^Q] = \mathbb{E} \left[ X_t \prod_{i=1}^{n-1} (1 + r\Delta + \sigma_i \Delta B_i) (1 + r\Delta + \sigma_n \Delta B_n) \right]
\]

\[
= \mathbb{E} \left[ X_{n-1}^Q (1 + r\Delta + \sigma_n \Delta B_n) \right] = \mathbb{E} \left[ X_{n-1}^Q (1 + r\Delta) \right] + \mathbb{E} \left[ X_{n-1} \sigma_n \Delta B_n \right]
\]

\[
= \mathbb{E} \left[ X_{n-1}^Q (1 + r\Delta) \right] + \mathbb{E} \left[ X_{n-1} \sigma_n \right] \mathbb{E} \Delta B_n = \mathbb{E} \left[ X_{n-1}^Q (1 + r\Delta) \right] = \cdots = \mathbb{E} \left[ X_t \prod_{i=1}^n (1 + r\Delta) \right] = \mathbb{E} \left[ X_t \right] (1 + r\Delta)^n.
\]

Finally, observing that \( \Delta = \frac{T-t}{n} \), we will have

\[
\mathbb{E}^Q[X_T] = \mathbb{E}[X_t] \left( 1 + \frac{r(T-t)}{n} \right)^n \to \mathbb{E}[X_t] e^{r(T-t)},
\]

when \( n \) goes to infinity, as stated. \( \Box \)
Formula (6.2) was verified computationally as part of the simulations of section 7. Also the estimates for the call option prices were given by

\[
\text{Call}_{T,K} = e^{-r(T-t)} \frac{1}{K} \sum_{k=1}^{K} (X_{T,k}^Q - K)_+ 
\]

with \(X_{T,k}^Q, k = 1, \ldots, K\) a set of simulated independent values.

7. Benchmarking

We present next the comparison results of pricing in the Black-Scholes model and in the model proposed. The data set used for this study is the following. We selected three stocks for which the graphic representation conveys evidence of regimes, namely Hewlet-Packard, Adobe Systems and Procter and Gamble. For these stocks we gathered call options data at date November 13 2009, consisting of prices (last, bid and ask) as well as volume for each expire dates and the available strike prices.

For the purpose of comparing the two calculation methods, we estimated the historical volatility of the price data by the usual standard deviation of price returns and also estimated the volatility process parameters in the proposed model. With the historical volatility we determine the Black-Scholes price by Black-Scholes formula. With estimated volatility process parameters we calculated, by Monte-Carlo simulation, the proposed model prices. We next compare the calculated prices with the real prices (last, bid and ask) with the following comparison criteria.

**Cr-1** The relative error between calculated \(P_c\) and market prices (last, bid and ask) \(P_m\), given by \((P_c - P_m)/P_m\). For each call option expire date we summed the relative errors in each available strike price. When the result is favorable to the proposed model against the Black-Scholes model it is printed in red.

**Cr-2** For each call option expire date, the number of times that the proposed calculation method outperformed the Black Scholes calculation method measured by the relative error as defined in Cr-1. We then checked if the result was statistically signficative by considering the null hypothesis \(H_0 : p \leq 0.5\) with \(p\) being the probability of success of a binomial trial with number of observations equal to the number of different strike prices and number of successes equal to the number of times that the proposed calculation method outperformed the Black Scholes calculation method. When the result is statistically significative up to a significance level of 0.05 we printed it in red. In the second row of this criterion we printed the p-value of the observed number of successes.

**Cr-3** For each call option expire date and for the strike price closest to the last observed stock price, the relative error as defined in Cr-1. When the result is favorable to the proposed model against the Black-Scholes model it is printed in red.

We present also another set of criteria more related on the real business value of the model proposed. We compare the cash-flows generated by buying a call option and possible exercising it at the expire date considering three option acquisition

\footnote{All the computational (Mathematica\textsuperscript{TM} and Excel\textsuperscript{TM}) files used in this section are downloadable at http://ferrari.dmat.fct.unl.pt/personal/mle/pps/pm-mle2009a.html.}
Table 3. Benchmarking against the Black-Scholes model: relative error criteria.

<table>
<thead>
<tr>
<th>Call option expire date: November 20, 2009</th>
<th>HP</th>
<th>Adobe</th>
<th>PG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices</td>
<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
</tr>
<tr>
<td>Cr-1 BS</td>
<td>6.158 4.597 5.392</td>
<td>6.158 4.597 5.392</td>
<td>4.050 4.050 3.208</td>
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<tr>
<td>Cr-1 p-v</td>
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<td>0.419 0.065 0.076</td>
<td>0.227 0.109 0.227</td>
</tr>
<tr>
<td>Cr-2 BS</td>
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<td>0.236 0.339 0.236</td>
<td>0.252 0.303 0.206</td>
</tr>
<tr>
<td>Cr-2 DTRS</td>
<td>0.236 0.339 0.236</td>
<td>0.236 0.339 0.236</td>
<td>0.252 0.303 0.206</td>
</tr>
</tbody>
</table>

<table>
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<th>Call option expire date: December 18, 2009</th>
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<th>PG</th>
</tr>
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<td>Prices</td>
<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
</tr>
<tr>
<td>Cr-1 BS</td>
<td>6.014 8.618 5.830</td>
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<td>4.488 4.488 3.996</td>
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<td>0.002 0 0</td>
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<td>Cr-2 BS</td>
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<td>0.385 0.385 0.282</td>
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<td>Cr-1 BS</td>
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<td>10.499 6.250 5.830</td>
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<td>0.372 0.456 0.372</td>
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<td>Cr-2 DTRS</td>
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<td>Last Bid Ask</td>
</tr>
<tr>
<td>Cr-1 BS</td>
<td>10.499 6.250 5.830</td>
<td>10.499 6.250 5.830</td>
<td>11.144 4.741 5.119</td>
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<tr>
<td>Cr-1 p-v</td>
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<td>0.076 0 0</td>
<td>0.008 0.008 0.008</td>
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<tr>
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<td>0.372 0.456 0.372</td>
<td>0.318 0.349 0.289</td>
</tr>
<tr>
<td>Cr-2 DTRS</td>
<td>0.372 0.456 0.372</td>
<td>0.372 0.456 0.372</td>
<td>0.318 0.349 0.289</td>
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<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
</tr>
<tr>
<td>Cr-1 BS</td>
<td>5.797 7.502 5.780</td>
<td>5.797 7.502 5.780</td>
<td>4.547 4.388 3.416</td>
</tr>
<tr>
<td>Cr-1 p-v</td>
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<td>0.000 0 0</td>
<td>0.000 0 0</td>
</tr>
<tr>
<td>Cr-2 BS</td>
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<td>0.175 0.256 0.283</td>
<td>0.423 0.423 0.353</td>
</tr>
<tr>
<td>Cr-2 DTRS</td>
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<td>0.175 0.256 0.283</td>
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<td>Last Bid Ask</td>
<td>Last Bid Ask</td>
</tr>
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<td>Cr-1 BS</td>
<td>7.353 8.336 6.742</td>
<td>7.353 8.336 6.742</td>
<td>4.099 3.959 3.746</td>
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<tr>
<td>Cr-1 DTRS</td>
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<td>6.166 7.069 5.662</td>
<td>4.099 3.959 3.746</td>
</tr>
<tr>
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<td>0.099 0.099 0.099</td>
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<tr>
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<td>0.199 0.256 0.225</td>
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<tr>
<td>Cr-2 DTRS</td>
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<td>0.199 0.256 0.225</td>
<td>0.468 0.541 0.433</td>
</tr>
</tbody>
</table>
Cr-5 For each call option expire date and for the strike price closest to the last stock price, the proposed method cash-flow, the Black-Scholes cash-flow and the market cash-flow generated by the reported price. If the result is favorable to the proposed model against the Black-Scholes model it is printed in red.

Cr-6 For each call option expire date, the number of times that the proposed method cash-flow outperformed the Black-Scholes cash-flow and the market cash-flow generated by the reported price and, in the third column under each stock, the number of times that the Black-Scholes cash-flow outperformed the market cash-flow generated by the reported price. As in the second criterion above, if the number is statistically significative up to a significance level of 0.05% it is printed in red and the correspondent p-value is printed in the second row.

Table 4. Benchmarking against the Black-Scholes model: cash-flow criteria.

<table>
<thead>
<tr>
<th>Call option expire date: November 20, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>HP</td>
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<tr>
<td>Cr-4 Sum c-f</td>
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<td>Cr-6 #</td>
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<td>p-v</td>
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<tbody>
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<td>HP</td>
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</tr>
<tr>
<td>Cr-5 c-f</td>
</tr>
<tr>
<td>Cr-6 #</td>
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<tr>
<td>p-v</td>
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<tr>
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<tbody>
<tr>
<td>HP</td>
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<tr>
<td>---</td>
</tr>
<tr>
<td>Cr-4 Sum c-f</td>
</tr>
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<td>Cr-5 c-f</td>
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<td>p-v</td>
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8. Conclusions

We introduced and studied a SDE model, useful for the price evolution of stocks, by dividing the phase space in two regions and considering that the solution process follows, in each region, a different diffusion. The solution process initially follows the diffusion corresponding to the region to which the initial condition belongs, until the trajectory crosses the threshold from one phase space region to the other. After the delay has elapsed, the process follows the diffusion corresponding to the second regime. If the process crosses again the threshold then, after the delay, it starts following again the diffusion corresponding to the initial regime.

We developed an efficient procedure for the estimation of all the parameters of the model (diffusion parameters for the two regimes, threshold and delay) in the particular case in which the SDE defining the diffusion, in each region, corresponds to a geometric Brownian motion SDE with, possibly, a random initial condition. We showed, for the general case, that if both the classification of the observations in regimes and the delay are known, then we can define consistent estimators for the threshold. With a simulation study we showed that the estimation procedure gives satisfactory results.
We applied the estimation procedure to a set of 21 stocks of the NYSE, observing that in some of these stocks there exists fairly differentiated regimes. Using two sets of criteria, we benchmarked the model introduced against the usual Black-Scholes geometric Brownian motion model for three stocks for which the regimes are clearly differentiated. In contrast with the Black-Scholes model the proposed model allows for the historic volatility to take two different values, each one corresponding to a different regime, thus permitting a better statistical adjustment. Accordingly the results are very satisfactory, even in the case of P&G where the two regime volatilities are not statistically different.

References


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Figure 2. Estimated regimes and thresholds for various stocks; data range from January 2005 to November 2009 (Philip Morris excepted); the two colors differentiate the classification of each observation in one of the two regimes.