ROOT’S BARRIER, VISCOSITY SOLUTIONS OF OBSTACLE PROBLEMS AND REFLECTED FBSDES

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Abstract. Following work of Dupire (2005), Carr–Lee (2010) and Cox–Wang (2011) on connections between Root’s solution of the Skorokhod embedding problem, free boundary PDEs and model-independent bounds on options on variance we propose an approach with viscosity solutions. Besides extending the previous results, it gives a link with reflected FBSDEs as introduced by El Karoui et. al. (1997) and allows for easy convergence proofs of numerical schemes via the Barles–Souganidis (1991) method.

1. Introduction

Let $B$ be standard, real-valued Brownian motion (started at 0) and $\mu$ a centered integrable probability measure on $(\mathbb{R}, B(\mathbb{R}))$. In 1961 Skorokhod [36] asked whether it is possible to find a stopping time $\tau$ such that $B_\tau \sim \mu$ and $B^\tau = (B_{t\wedge \tau})_t$ is uniformly integrable; a stopping time $\tau$ with these properties is now usually referred to as a solution of the Skorokhod embedding problem or Skorokhod stopping problem. Further, he provided an affirmative answer by constructing a stopping time which depends on an additional random variable independent of $B$. Skorokhod’s motivation came from invariance properties of random walks and subsequently this question attracted the interested of many researchers and a large number of different approaches and generalizations were developed, each approach giving rise to a different stopping time which solves the same Skorokhod embedding problem — we do not even attempt to give a brief summary but refer to the surveys [27, 28, 21] which list many different approaches and discusses several applications.

One of the earliest approaches after Skorokhod was initiated by Root in 1969 [30] who showed that if $\mu$ is centered and has a second moment, then it is possible to find a subset $R$ of $[0, \infty) \times [−\infty, \infty)$, the so-called Root barrier, such that its first hitting time by the time–space process $(t, B_t)$, $\tau_R = \inf \{ t \geq 0 : (t, B_t) \in R \}$, solves the Skorokhod embedding problem. In addition to this intuitive solution, Loynes [26] introduced extra conditions under which Root’s barrier $R$ is uniquely defined (as a subset of $[0, \infty) \times [−\infty, \infty)$) and Rost [34] proved that among all stopping times $\tau$ which solve the Skorokhod embedding problem, Root’s solution minimizes $E[\tau^2]$; further he gave important generalizations and connections with potential theory, characterized the possible stopping distributions of Markov processes by introducing a filling scheme in continuous time and proved the existence of another barrier which leads to a hitting time which maximizes $E[\tau^2]$. Despite all these nice properties of Root’s and Rost’s approach they had until very recently the significant disadvantage that it was not known how to calculate the barriers — in fact, only for a handful of very simple target measures $\mu$ the explicit form of the barriers was known.

One of the more recent motivations to study the Skorokhod embedding problem comes from mathematical finance where Skorokhod embeddings naturally appear if one is interested in model-independent prices of exotic derivatives, i.e. “extremal” solutions of the Skorokhod problem lead to robust lower and upper bounds for arbitrage-free prices of exotics — we refer to the survey’s of Hobson [21] and Obłój [28]. Motivated by this, Dupire [14] gave in 2005 a presentation where he firstly pointed out that in the case of options on variance these extremal solutions are the ones of Root [30] resp. Rost [34] and, secondly, that these barriers can be calculated by solving nonlinear parabolic PDEs, namely solutions to free boundary problems. This was further developed by Carr–Lee [8] and in 2011 Cox–Wang [11] made this connection precise using the variational approach as developed in the 1970’s by A. Bensoussan–J.L. Lions et. al. [6] (and do much more — we refer the reader to [11]). This article is inspired by all these beautiful results and our goal is to study the connection of Root’s barrier with the parabolic obstacle problem from the perspective of viscosity theory; more precisely, given two probability measures $\mu, \nu$ on $(\mathbb{R}, B(\mathbb{R}))$ and a function $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we want to find (under appropriate conditions on $\sigma, \mu, \nu$) the Root barrier $R$ such that its first hitting time $\tau_R$ solves the following Skorokhod embedding problem

\[
\begin{align*}
\{ \quad dX_t & = \sigma(t, X_t) \, dB_t, \quad X_0 \sim \mu, \\
X_{\tau_R} & \sim \nu \quad \text{and} \quad X^\tau = (X_{t \wedge \tau})_t \quad \text{is uniformly integrable.} 
\end{align*}
\]
by identifying $R$ with a continuous function $u \in C((0,\infty) \times \mathbb{R}, \mathbb{R})$ which is a viscosity solution of

$$\min \left( u + \int_{\mathbb{R}} |x| \nu(dx), \frac{\partial u}{\partial t} - \frac{\sigma^2}{2} \Delta u \right) = 0 \text{ on } (0,\infty) \times \mathbb{R},$$

$$u(0,. ) = - \int_{\mathbb{R}} |x| \mu(dx) \text{ on } \mathbb{R}.$$  \hspace{1cm} (2)

and with the quadruple of process $(X,Y,Z,K)$, solution to the reflected Forward–Backward SDE (for $h(\cdot) := - \int_{\mathbb{R}} |y| \nu(dy)$, $T > 0$, $(t,x) \in [0,T] \times \mathbb{R}$ and $s \in [t,T]$)

$$X_{s,t}^{t,x} = x + \int_{t}^{s} \sigma(T-r,X_r) dW_r,$$

$$Y_{s,t}^{t,x} = u \left(0, X_{T}^{t,x} \right) + K_{s,t}^{t,x} - K_{t,t}^{t,x} - \int_{T}^{s} Z_{r}^{t,x} dW_r,$$

$$Y_{s,t}^{t,x} \geq h(X_{s}^{t,x}), t < s \leq T \text{ and } \int_{t}^{T} (Y_{s,t}^{t,x} - h(X_{s}^{t,x})) dK_{s,t}^{t,x} = 0.$$  

What we consider among the benefits of the viscosity approach are that it

- covers the case when $\sigma$ is time dependent and only elliptic, i.e. no time-homogeneity or uniform ellipticity assumptions as in [11] are needed (especially the important case for model-independent bounds when $\sigma(t,x) = x$ is immediately in our setting without an extra transformation),
- gives a direct link to reflected FBSDEs and their Feynman–Kac representation via classic work of El Karoui, Kapoudjian, Pardoux, Peng and Quenez [15],
- uses continuous functions instead of values in weighted Sobolev spaces and allows for short proofs of convergence of numerical schemes via the Barles–Souganidis method.

Further, we introduce a generalization of Loyays’ notion of regular barriers, so-called $(\mu, \nu)$-regular barriers which allows to establish a one–to–one correspondence between the problems (1) and (2).

In Section 2 we recall several results from the literature and discuss the general Root embedding problem, in Section 3 we recall several results in viscosity theory and the connection with reflected FBSDES. Section 4 contains our main result: the link between the embedding problem, PDE in viscosity sense and reflected FBSDE. In Section 5 apply the Barles–Souganidis [3] method in this context and we finish in Section 6 by recalling the relevance of Root’s barrier for the derivation of bounds on variance options.

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2. Convex order, potential functions, root barriers and root embedding

Already for the case of Brownian motion one cannot hope to find Skorokhod embeddings for two arbitrary probability measures $\mu$ and $\nu$, i.e. $B_0 \sim \mu$, $B_\tau \sim \nu$. Intuitively, a necessary condition is that the target distribution $\nu$ should have a higher dispersion than the initial distribution $\mu$. This can be expressed by a classic order relation on the set of probability measures, namely the so-called convex order. In the context of Root solution this order relation appears also naturally when one thinks in terms of the potential theoretic picture as developed in this context by Rost [31, 32, 33, 34], Chacon [10] and others.

Definition 1. Let $k \in \mathbb{N}$, $m \in \mathbb{R}$ and denote with $\mathcal{M}^k_m$ the set of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with a finite $k$-th moment and mean $m$ (i.e. $\mu \in \mathcal{M}^k_m$, iff $\int_\mathbb{R} x \mu(dx) = m$ and $\int_\mathbb{R} |x|^k \mu(dx) < \infty$) and set $\mathcal{M}^k = \bigcup_{m \in \mathbb{R}} \mathcal{M}^k_m$, $\mathcal{M} = \bigcup_{k \in \mathbb{N}[0]} \mathcal{M}^k$.

Definition 2. Let $m \in \mathbb{R}$. A pair of probability measures $(\mu, \nu) \in \mathcal{M}^1 \times \mathcal{M}^1$ is in increasing convex order, $\mu \leq_{cx} \nu$, if

$$\int_{\mathbb{R}} f(x) \mu(dx) \leq \int_{\mathbb{R}} f(x) \nu(dx)$$

for every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$.

(provided the integrals exist). We also say that two measures $\mu, \nu$ are in convex order if either $\mu \leq_{cx} \nu$ or $\nu \leq_{cx} \mu$.

Two real-valued random variables $X,Y$ are in increasing convex order, denoted $X \leq_{cx} Y$, if $\mathcal{L}(X) \leq_{cx} \mathcal{L}(Y)$.

Remark 1. Assume $\mu \in \mathcal{M}^1_m$, $\nu \in \mathcal{M}^1_n$ and $\mu \leq_{cx} \nu$. Then $m = n$, this follows from choosing $f(x) = x$ and $f(x) = -x$.

The intuition being that $\mu \leq_{cx} \nu$ signifies that a $\mu$ distributed random variable is less likely to take very large values than a $\nu$ distributed random variable since extremal values of convex functions are obtained on intervals of the form $(-\infty, a) \cup (b, \infty)$. As it turns out, an especially important case is given when (3) is applied with $f(.) = |. - m|$ resp. $f(.) = -|. - m|$ for fixed $m$. 

\footnote{Sometimes this is also called the Choquet order.}
Definition 3 (Potential function). Associate with every \( \mu \in \mathcal{M}^{1} \) a continuous non-positive function \( u_{\mu} \in C(\mathbb{R}, (-\infty, 0]) \) given as

\[
u(\mu)(x) := -\int_{\mathbb{R}} |x-y| \mu(dy).
\]

We say that \( u_{\mu} \) is the potential function of the probability measure \( \mu \).

Example 1. Denote with \( \delta_{m} \) the Dirac measure at \( m \in \mathbb{R} \). Then \( u_{\delta_{m}}(x) = -|x-m| \) and it is easy to verify that \( u_{\mu} \leq u_{\delta_{m}} \) for every \( \mu \in \mathcal{M}^{1}_{m} \), resp. \( \mu \geq \delta_{m} \).

Lemma 1. Let \( m \in \mathbb{R} \) and \( \mu, \nu \in \mathcal{M}^{1}_{m} \). Then the following are equivalent

1. \( \mu \leq \nu \)
2. \( -\int (y-x)^{+} \mu(dy) \geq -\int (y-x)^{+} \nu(dy) \), \( \forall x \in \mathbb{R} \)
3. \( u_{\mu}(x) \geq u_{\nu}(x) \), \( \forall x \in \mathbb{R} \)

Moreover, \( \mu \leq \nu \) implies that \( \int_{\mathbb{R}} x\mu(dx) = \int_{\mathbb{R}} x\nu(dx) \) and if \( \mu, \nu \in \mathcal{M}^{2}_{m} \) then \( \int_{\mathbb{R}} x^{2}\mu(dx) \leq \int_{\mathbb{R}} x^{2}\nu(dx) \).

Proof. Follows from [35, Section 3.A].

Before we justify the name “potential function” let us note that the classic connection between positive [non-negative] Radon measures and second derivatives of concave [convex] functions reads in this setting as

Proposition 1. There exists a one-to-one correspondence between potential functions and probability measures.

More precisely:

1. Let \( \mu \in \mathcal{M}^{1}_{m} \) for some \( m \in \mathbb{R} \). Then \( u_{\mu}(x) = -\int_{\mathbb{R}} |x-y| \mu(dy) \) is a concave function satisfying

\[
u(\mu)(x) = -|x-m| \text{ and } \lim_{x \to \pm \infty} u_{\mu}(x) - u_{\delta_{m}}(x) = 0
\]

2. Let \( u : \mathbb{R} \to \mathbb{R} \) be a concave function satisfying

\[
u(u)(x) \leq -|x-m| \text{ and } \lim_{x \to \pm \infty} u(x) + |x-m| = 0
\]

for some \( m \in \mathbb{R} \). Then there exists a probability measure \( \mu \in \mathcal{M}^{1}_{m} \) such that \( u \) is the potential function of \( \mu \), i.e. \( u = u_{\mu} \).

Moreover, the connection is given by the relation \( \mu = \frac{1}{2}u'' \) in the sense of distributions.

Proof. See [20, Proposition 2.1] and [5, Proposition 4.1].

Remark 2. To explain the name “potential function” we briefly recall the classic potential theory for Markov processes (cf. [7, 27, 28] for details): consider a real-valued Brownian motion \( B \) and denote its semigroup operator with \( (P_{t}^{B}) \). The potential kernel is defined as \( U^{B} = \int_{0}^{\infty} P_{t}^{B} dt \), i.e. \( U^{B} \) can be seen as a linear operator on the space of measures on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) by setting \( \mu U^{B} = \int_{0}^{\infty} \mu P_{t}^{B} dt \) which is of course nothing else than the occupation measure along Brownian trajectories started with \( B_{0} \sim \mu \). If \( \mu \) is a signed measure with \( \mu(\mathbb{R}) = 0 \) and finite first moment, then the Radon–Nikodym density with respect to the Lebesgue measure is given as

\[
u(U^{B}) = -\int |x-y| \mu(dy).
\]

Since (in dimension one) Brownian motion is recurrent, \( U^{B} \) is infinite if \( \mu \) is positive. However, the right hand side \( -\int |x-y| \mu(dy) \) is still well defined for every \( \mu \in \mathcal{M}^{1} \) and Chacon [10] demonstrated that this is indeed a very useful quantity in the study of Martingales and Markov processes with respect to hitting times.

We summarize some properties of the potential functions of probability measures which we use throughout. The proofs are standard and can be found in [27, Proposition 2.3] and [28, Section 3.2].

Proposition 2. Let \( \mu, \nu \in \mathcal{M}^{1} \). Then

1. \( \mu \leq \nu \) iff \( \mu \geq \nu \) iff \( \mu(\cdot) \geq \nu(\cdot) \),
2. \( u_{\mu} \) is concave and Lipschitz continuous with Lipschitz constant equal to one,
3. if \( \mu(\mathbb{R}) = m \) then \( u_{\mu}(x) \leq u_{\delta_{m}}(x) = -|x-m| \),
4. if \( \nu \in \mathcal{M}^{1} \) then \( \forall b \in \mathbb{R} \) we have

\[
u(\mu)(b, \infty) = u_{\nu}(b, \infty) \text{ iff } \mu(\cdot) \equiv \nu(\cdot) \text{ on } (b, \infty)
\]

5. if \( \mu_{n} \in \mathcal{M}^{1} \), \( \mu_{n} \to_{\nu} \mu \) weakly if and only if \( u_{\mu_{n}} \) converges for some point \( x_{0} \in \mathbb{R} \) then \( u_{\mu_{n}} \) converges for all \( x \in \mathbb{R} \) and \( \exists c \geq 0 s.t. \lim_{n \to \infty} u_{\mu_{n}}(x) = u_{\mu}(x) + c \). Further, if \( \forall \nu \in \mathcal{M}^{1} \) such that \( u_{\mu_{n}} \geq u_{\nu} \) then \( \mu \in \mathcal{M}^{1} \) and \( c = 0 \). Conversely, if \( u_{\mu_{n}} \to_{\nu} u_{\nu}(x) \forall x \in \mathbb{R} \) for some \( \mu_{n}, \mu \in \mathcal{M}^{1} \) then \( \mu_{n} \to_{\nu} \mu \) weakly,
6. if \( \mu \leq \nu \) then \( \lim_{x \to x_{0}} (u_{\mu}(x) - u_{\nu}(x)) = 0 \),
7. \( u_{\mu} \) is almost everywhere differentiable on \( \mathbb{R} \) and its right derivative at \( x \) equals \( 1 - 2\mu((-\infty, x]) \) and its left derivative equals \( 1 - 2\mu((x, \infty)) \).

2The symbol “\( \Rightarrow \)” denotes weak convergence of probability measures.
Since we want to embed a measure via a Markov process, the potential function which is given by the image measure of this Markov process is a fundamental quantity.

**Definition 4.** If \( X \) is a real-valued, continuous Markov process carried on a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) with \( \mathbb{P} \circ X^{-1}_t \in \mathcal{M}^1 \) for every \( t \geq 0 \) then we call

\[
u_X(t, x) := u_{pX^{-1}}(x)
\]

the potential function of \( X \) (at time \( t \)).

As shown by Chacon [10, Lemma 3.1a], potential functions give an intuitive characterization of Martingales.

**Proposition 3.** Let \( X \) be a real-valued, adapted process. Then \( X \) is a martingale iff for every pair \( \tau_1, \tau_2 \) of bounded stopping times with \( \tau_1 \leq \tau_2 \),

\[
u_X(\tau_1, \cdot) \geq \nu_X(\tau_2, \cdot).
\]

### 2.1. Root's solution and classic results on barriers

Our aim is to provide explicit solutions of the Skorokhod embedding problem not only for Brownian motion but for diffusion martingales.

**Assumption 1.** \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is a filtered probability space that satisfies the usual conditions and carries a standard Brownian motion \( B \). Assume that \( \sigma \in C([0, \infty) \times \mathbb{R}, \mathbb{R}) \) is Lipschitz in space and of linear growth, both uniformly in time, that is

\[
s_{Lip} := \sup_{t \in [0, \infty)} \sup_{x, y} \frac{|\sigma(t, x) - \sigma(t, y)|}{|x - y|} < \infty \quad \text{and} \quad s_{LG} := \sup_{t \in [0, \infty)} \sup_{x \in \mathbb{R}} \frac{|\sigma(t, x)|}{1 + |x|} < \infty.
\]

Further, let \( m \in \mathbb{R}, (\mu, \nu) \in M^2_{\text{mm}}, \mu \ll \nu, \) and \( X_0 \sim \mu, X_0 \) independent of the \( \sigma \)-algebra generated by the Brownian motion \( B \) and assume that with probability one \( \sigma^2(t, X_t) > 0 \forall t \geq 0 \) where \( X = (X_t) \) denotes the unique strong solution of the SDE (6), i.e. the real-valued, local martingale \( X \) such that

\[
X_t = X_0 + \int_0^t \sigma(t, X_t) \, dB_t \quad \mathbb{P} - a.s
\]

Further, assume that for every \( t > 0, X_t \) has a continuous (but not necessarily smooth) density with respect to Lebesgue measure.

**Definition 5** (Skorokhod embedding problem). Let \((\sigma, \mu, \nu)\) and \((\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), \mathbb{P})\) be as in Assumption 1. If there exists a stopping time \( \tau \) (wrt the filtration \( (\mathcal{F}_t) \)) such that

\[
X_\tau \sim \nu, X^{\tau} = (X_{t \wedge \tau})_{t \geq 0}
\]

then we say that \( \tau \) solves the Skorokhod embedding problem given by \((\sigma, \mu, \nu)\). We denote with \( S(\sigma, \mu, \nu) \) the (possibly empty) set of stopping times \( \tau \) which solve the Skorokhod embedding problem given by \((\sigma, \mu, \nu)\).

**Remark 3.** Unless stated otherwise, all stopping times are stopping times in the filtration \( (\mathcal{F}_t) \).

As pointed out in the introduction there are many different solutions of the embedding problem, that is in general the set \( S(\sigma, \mu, \nu) \) can contain much more than one stopping time (in the survey article of Obloj [28] at least 20 different solutions are given for embedding laws in Brownian motion, i.e. \( (\sigma, \mu, \nu) = (1, \delta_0, \nu) \)). In terms of applications a natural requirement is to consider stopping times which are minimal.

**Definition 6.** A stopping time \( \tau \) is called minimal for the process \( X \) if for every other stopping time \( \rho \) such that \( \rho \leq \tau \) and \( X_{\tau} \sim X_{\rho} \) it follows that \( \rho = \tau \).

This article is focused on the approach initiated by Root in 1968 [30] and further developed by Rost, Loynes, Kiefer et. al [34, 32, 31, 25, 26] of solving Skorokhod embeddings by stopping times which are the first entrance times of the time-space process \((t, X_t)_{t \geq 0}\) into a closed subset \( R \) of \([0, \infty] \times [-\infty, \infty] \).

**Definition 7.** A closed subset \( R \) of \([0, \infty] \times [-\infty, \infty] \) is a Root barrier \( R \) if

1. \((t, x) \in R \) implies \((t + r, x) \in R \forall r \geq 0,\)
2. \((\pm \infty, x) \in R \forall x \in [-\infty, \infty],\)
3. \((t, \pm \infty) \in R \forall t \in [0, +\infty].\)

We denote by \( \mathcal{R} \) the set of all Root barriers \( R \). Given \( R \in \mathcal{R} \), its barrier function \( f_R : [-\infty, \infty] \to [0, \infty] \) is defined as

\[
f_R(x) := \inf \{ t \geq 0 : (t, x) \in R \}, \quad x \in [-\infty, \infty].
\]

Barrier functions have several nice properties such as being lower semi-continuous and that \((f_R(x), x) \in R \) for any \( x \in \mathbb{R} \) (see [26, Proposition 3] for more properties).
Definition 8. We denote with $\mathcal{R}(\sigma, \mu, \nu)$ as the (possibly empty) subset of $\mathcal{R}$ of all $R \in \mathcal{R}(\sigma, \mu, \nu)$ such that $\tau_R \in S(\sigma, \mu, \nu)$ where $\tau_R$ is the first hitting time of $R$ by $(t, X_t)$, i.e.

$$\tau_R = \inf \{ t > 0 : (t, X_t) \in R \}.$$ 

We also say that $R \in \mathcal{R}(\sigma, \mu, \nu)$ embeds the law $\mu$ into $\nu$ (via $X$) and call $\tau_R$ the Root stopping time.

Before we discuss conditions on $\sigma, \mu, \nu$ which imply the existence of a Root barrier, i.e. that $\mathcal{R}(\sigma, \mu, \nu) \neq \emptyset$, we note that simple examples show that the Root barrier is in general not unique, i.e. $|\mathcal{R}(\sigma, \mu, \nu)| > 1$ already for generic situations.

Example 2. The set $R = [0, \infty] \times [1, \infty] \cup [0, \infty] \times [-\infty, -1]$ is an element of $\mathcal{R}(1, \delta_0, \frac{1}{2}(\delta_{-1} + \delta_1))$, i.e. $R$ embeds the sum of two Diracs into a standard Brownian motion but obviously one can define $\tau_R$ differently, in fact any other Root barrier with a barrier function which coincides with $f_R$ on $[-1, 1]$ is also an element of $\mathcal{R}(1, \delta_0, \frac{1}{2}(\delta_{-1} + \delta_1))$.

This non-uniqueness problem was in the Brownian case resolved by Loynes [26] in 1970 who introduced the notion of regular Root barriers.

Definition 9 (Loynes [26] regularity). Let $R \in \mathcal{R}$. We say that $R$ resp. its barrier function $f_R$ is Loynes-regular if $f_R$ vanishes outside the interval $[x_R^-, x_R^+]$, where $x_R^-$ and $x_R^+$ are the first positive resp. first negative zeros of $f_R$. Given $Q, R \in \mathcal{R}$ we say that $Q, R$ are Loynes-equivalent if $Q \equiv f_R$ on $[x_Q^-, x_Q^+]$ and $[x_R^-, x_R^+]$.

Remark 4. If $Q, R$ are Loynes-equivalent then $x_R^+ = x_Q^+$ and $x_R^- = x_Q^-$. 

We advise the reader to be suspicious about the word “regular” since Loynes-regular Root barriers can have spikes (see examples in Section 2.2). For the case $(\sigma, \mu, \nu) = (1, \delta_0, \nu)$ existence and uniqueness of Loynes-regular Root barriers are standard.

Theorem 1 ([30, 26]). For every $\nu \in \mathcal{M}_2^\nu$ the set $\mathcal{R}(1, \delta_0, \nu)$ is not empty and there exists exactly one element in $\mathcal{R}(1, \delta_0, \nu)$ which is Loynes-regular. The second moment of the Root stopping time, $E[\tau_R^2]$ for $R \in \mathcal{R}(1, \delta_0, \nu)$, is minimal among all elements of $\mathcal{S}(1, \delta_0, \nu)$.

An advantage of Root’s solution besides its intuitive appeal (the stopping time being a hitting time) the property that $\tau_R$ minimizes $E[\tau^2]$ among all solutions of the Skorokhod embedding problem, i.e. in our notation $E[\tau_R^2] = \inf E[\tau^2]$ where the inf is taken over all $\tau \in \mathcal{S}(\sigma, \mu, \nu)$. This was proven in Root [34] (as well as in [11]) and is the very reason why Root’s solution is important for financial applications, see Section 6.

2.2. Regularity beyond Loynes. We now extend the discussion on Root barriers to our more general setting of Root embeddings for Itô-martingales with random initial distribution, i.e. we discuss $\mathcal{R}(\sigma, \mu, \nu)$. To that end we need to adapt several results of Loynes [26]. First, note that already in the Brownian case $(\sigma \equiv 1)$ we need a modification of the notion of Loynes-regularity, a notion that is tailor made for the Dirac as initial distribution as the example below shows.

Example 3. Let $\mu = \frac{1}{2}(\delta_2 + \delta_{-2})$ and $\nu = \frac{1}{3}(\delta_3 + \delta_1 + \delta_{-1} + \delta_{-3})$. It is easy to see by symmetry properties of the Brownian motion that for $a = b = 0$ the barrier

$$Q_{a,b} = [0, \infty] \times [3, \infty] \cup [0, \infty] \times \{1\} \cup [0, \infty] \times \{-1\} \cup [0, \infty] \times [-3, \infty] \cup \{+\infty\} \times [-\infty, +\infty]$$

is an element of $\mathcal{R}(1, \mu, \nu)$, as is

$$R = [0, \infty] \times [3, \infty] \cup [0, \infty] \times \{1\} \cup [0, \infty] \times \{-1\} \cup [0, \infty] \times [-3, \infty] \cup \{+\infty\} \times [-\infty, +\infty].$$

However, neither is Loynes-regular and in fact there cannot exist a Loynes-regular barrier in $\mathcal{R}(1, \mu, \nu)$. (Assume $Q_{a,b} \in \mathcal{R}(1, \mu, \nu)$, then $a, b > 0$ otherwise it would not be Loynes regular; now note that $Q_{0,0} \in \mathcal{R}(1, \mu, \nu)$, hence every other $Q_{a,b}$ puts under $\nu$ more mass on the point $3$ than the required $\frac{1}{4}$ since the geometry of $Q_{a,b}$ implies that only more trajectories can hit the line $[0, \infty] \times \{3\}$ than in the case $a = b = 0$; further, every element of $\mathcal{R}(1, \mu, \nu)$ must coincide with $Q_{0,0}$ on $[0, \infty] \times \{1, 3\} \cup [0, \infty] \times [-3, -1]$).

Motivated by the above we introduce the notion of $(\mu, \nu)$-regular barriers.

Definition 10. For $\mu \leq \nu$ define

$$N^{\mu,\nu} := \{ x \in (-\infty, +\infty) : u_{\mu}(x) = u_{\nu}(x) \} \cup \{ \pm \infty \}$$

and $N^{\mu,\nu} = [0, +\infty] \times N^{\mu,\nu}$. 

A Root barrier $R$ is $(\mu, \nu)$-regular if $R = R \cup N^{\mu,\nu}$ or equivalently if $f_R(x) = 0 \ \forall x \in N^{\mu,\nu}$. Denote with $\mathcal{R}_{\text{reg}}(\sigma, \mu, \nu)$ the subset of $\mathcal{R}(\sigma, \mu, \nu)$ of $(\mu, \nu)$-regular Root barriers. Further, two Root barriers $R, Q$ are $(\mu, \nu)$-equivalent if $R \setminus (0, \infty) \times (N^{\mu,\nu})^0 = Q \setminus (0, \infty) \times (N^{\mu,\nu})^0$ or equivalently if $f_R(x) = f_Q(x) \ \forall x \in (N^{\mu,\nu})^c$.

3We denote with $\mathcal{A}$ the closure and with $A^c$ the interior of a given set $A$. 

If $F^\mu$ and $F^\nu$ are the cumulative distribution functions of $\mu$ and $\nu$ respectively, then the set $\{x : F^\mu(x) = F^\nu(x)\} \cup \{\pm \infty\}$. The next Lemma shows that in the case $\sigma \equiv 1$ and $\delta = 0$ the $(\mu, \nu)$-regularity of a Root barrier is equivalent to Loynes regularity.

**Lemma 2.** Let $\nu \in \mathcal{M}_2^\mu$ and $R \in \mathcal{R}(1, \delta_0, \nu)$. Then $R$ is Loynes-regular iff $R$ is $(\delta_0, \nu)$-regular.

**Proof.** Assume $R$ to be Loynes-regular, then $f_R(x) = 0$ for $\forall x \notin (x_0^R, x_0^R)$ where $x_0^R, x_0^R$ are as in Definition 9. Since the embedding is possible, i.e. $X_{\tau_0} \sim \nu$, then via the regularity of the barrier $R$ one has that $\nu([x_0^R, x_0^R]) = 1$, i.e. $\nu$ puts no mass outside $[x_0^R, x_0^R]$ (by definition so does $\delta_0$). In other words outside $[x_0^R, x_0^R]$ both measures $\nu$ and $\delta_0$ are the same which by point (4) of Proposition 2 implies that $u_{\delta_0}(x) = u_{\nu}(x)$ for $x \notin [x_0^R, x_0^R]$ and by continuity of the potential functions for any $x \notin [x_0^R, x_0^R]$. We conclude that $R$ is $(\delta_0, \nu)$-regular.

For the inverse direction, just remark that being $R$ a $(\delta_0, \nu)$-regular barrier we have due to $\delta_0 \leq c_\tau \nu$ that $N_{\delta_0, \nu} \cap \mathbb{R}$ has the form $N_{\delta_0, \nu} \cap \mathbb{R} = \mathbb{R} \setminus (a, b)$ for some $a < b < \nu$, in other words $f_R(x) = 0$ for any $x \notin (a, b)$. Notice that the convex order relation (with $\delta_0$) implies that if $u_{\delta_0}(c) = 0$ for some $c > 0$ then $u_{\nu}(x) = 0$ for any $x \geq c$ (and corresponding condition for the case $c < 0$), this implies that $a$ and $b$ are the first negative resp. positive zero of $f_R$. Hence $R$ is Loynes-regular. □

Definition 10 allows to extend Loynes’ [26, Theorem 1] to the present setting, i.e. if there exists an embedding with a Root barrier then there also exists an embedding with a $(\mu, \nu)$-regular Root barrier.

**Theorem 2.** Let $\mu, \nu \in \mathcal{M}^2_1$, $\mu \leq c_\tau \nu$ and let $X$ be a real-valued continuous square integrable local Martingale $X$ with a quadratic variation that is strictly increasing and both generate the same law. Notice that the convex order relation (with $\delta_0$) allows to extend Loynes’ [26, Theorem 1] to the present setting, i.e. if there exists an embedding with a $(\mu, \nu)$-regular Root barrier then there also exists an embedding with a $(\mu, \nu)$-regular Root barrier.

We prepare the proof with a lemma concerning the union and intersection of Root barriers.

**Lemma 3.** Take $\nu \in \mathcal{M}^2_1$ and let $Y$ be a process with continuous paths. Assume there exists $Q, R \in \mathcal{R}$ with first hitting times $\tau_Q$ and $\tau_R$ respectively and assume that $Y_{\tau_Q} \sim \nu$ and $Y_{\tau_R} \sim \nu$. Then $Q \cup R, Q \cap R \in \mathcal{R}$ and both generate the same law $\nu$ with respective stopping times $\min\{\tau_Q, \tau_R\}$ and $\max\{\tau_Q, \tau_R\}$.

**Proof.** This proof is a straightforward modification of [26, Proposition 4], we fill in some details and sketch the proof for the case of $Q \cup R$: let $f_R$ and $f_Q$ be the barrier functions of $R$ and $Q$ and define the set $K := \{x : f_Q(x) < f_R(x)\}$. Since both barriers (with their respective stopping times) generate $\nu$ we have that $P[\{Y_{\tau_Q} \in K\} = P[\{Y_{\tau_R} \in K\}]$ if

$$P[Y_{\tau_Q} \in K, Y_{\tau_R} \in K] = P[Y_{\tau_Q} \in K, Y_{\tau_R} \in K] + P[Y_{\tau_Q} \in K^c, Y_{\tau_R} \in K]$$

if $P[Y_{\tau_Q} \in K, Y_{\tau_R} \in K^c] = P[Y_{\tau_Q} \in K^c, Y_{\tau_R} \in K^c]$. A quick analysis shows that for a trajectory $Y(\omega)$ s.t. $Y_{\tau_Q}(\omega)$ ends in $K$ it is impossible that $Y_{\tau_R}(\omega) \notin K^c$: in other words, since the paths of $Y$ are continuous and in the set $[0, \infty) \times K$ the barrier $Q$ is to the left of $R$ then any path that hits $Q$ must hit $R$ before (hitting $Q$) and hence $P[Y_{\tau_Q} \in K^c, Y_{\tau_R} \in K] = 0 = P[Y_{\tau_Q} \in K, Y_{\tau_R} \in K^c]$. Finally, by decomposing the probability space in disjoint events (based on $K$ and $K^c$) and using the just obtained result one concludes $P[\{Y_{\tau_Q} \in K, Y_{\tau_R} \in K\} \cup \{Y_{\tau_Q} \in K^c, Y_{\tau_R} \in K^c\}] = 1$. The rest of the proof follows as in [26]. □

We can now give the proof of Theorem 2.

**Proof of Theorem 2.** To see that $Q$ is $(\mu, \nu)$-equivalent to a $(\mu, \nu)$-regular barrier note that since $u_{\nu} \leq u_{\mu}$ (and we embed by assumption) the continuous time-space process $(t \wedge \tau_Q, X_{t \wedge \tau_Q})$ does not enter $[0, \infty) \times N_{\mu, \nu}$, hence $R := Q \cup N_{\mu, \nu}$ is a also an element of $\mathcal{R}(\sigma, \mu, \nu)$, and moreover an element of $\mathcal{R}_{\operatorname{reg}}(\sigma, \mu, \nu)$.

To see the uniqueness, i.e. $[R_{\operatorname{reg}}(\sigma, \mu, \nu)] = 1$ suppose that there are two $(\mu, \nu)$-regular barriers $B, C$, each embedding $\nu$ (via $X$) with u.i. stopping times $\tau_B$ and $\tau_C$ respectively. Then $\Gamma = B \cup C$ also embeds $\nu$ with stopping time $\gamma = \min\{\tau_B, \tau_C\}$ (see Lemma 3); $\gamma$ is also u.i. Furthermore, since $X$ is a local martingale,

$$E[|X|_{\tau_B}] = E[X_{\tau_B}^2] = E[X_{\tau_C}^2] = E[|X|_{\gamma}]$$

Since the quadratic variation $t \rightarrow |X|_t$ is strictly increasing and $\gamma \leq \tau_B$ this already implies $|X|_{\gamma} = |X|_{\tau_B}$. The strict monotonicity of $t \rightarrow |X|_t$ then implies $\gamma = \tau_B$. The same arguments applies to $\tau_C$, hence we conclude that $\tau_B = \gamma = \tau_C$.

To see that $B, C$ and $B \cup C$ are the same (outside $N_{\mu, \nu}$ since in $N_{\mu, \nu}$ this must hold) we argue as in the proof of Lemma 2 in [26] by showing that if $B \neq \Gamma$ then also $\tau_B \neq \tau_C$. Note that $B \neq \Gamma$ implies that the existence of $x_0 \in \mathbb{R}$ such that $x_0 \notin N_{\mu, \nu}$ and (wlog) $f_B(x_0) < f_B(x_0)$ where $f_B(x_0) \neq 0$. Since the barriers are regular and lower semi-continuous functions attain their minimum on a compact set, $\exists \epsilon > 0 \text{ s.t. } f_B(x) > \epsilon$ on some neighborhood of $x_0$. Moreover $\epsilon$ can be chosen in such a way that $2\epsilon < f_B(x_0) - f_B(x_0)$, implying that a $\delta > 0$ exists s.th $f_B(x) > f_B(x_0) - \epsilon > f_B(x_0) + \epsilon$ for any $x$ in the set $\{x : |x - x_0| < \delta\}$. Since the embedding via $X$ is possible, there exists a positive probability that the time-space process $(t, X_t)_{t \geq 0}$ hits the line segment
2.3. The general Root embedding and optimality. We give a generalization of Theorem 1 showing that \( R(\sigma, \mu, \nu) \) is not empty, i.e. embedding \( \nu \in M^2 \) via the solution process of the SDE

\[
dX_t = \sigma(t, X_t) dB_t, \quad X_0 \sim \mu.
\]

Root’s original construction (for Brownian motion, i.e. \( \sigma \equiv 1 \) and \( \mu = \delta_0 \)) relies only on the continuity of the Brownian trajectories, the strong Markov property, that trajectories exit compact intervals in finite time and the fact that the process has no interval of constancy – hence (as remarked several times in the literature) the same construction generalizes. Here we give only the results, complete proofs are postponed to Appendix 7. It turns out to be natural for the proof to distinguish between the cases \( [X]_\infty = \infty \) and when \( X \) is a geometric Brownian motion (which implies only \( [\ln X]_\infty = \infty \)).

2.3.1. Existence and uniqueness of a Root barrier if \( [X]_\infty = \infty \). We now state a general result on the existence of the Root embedding via the solution of (6) (the proof is given in the appendix).

**Theorem 3** (Root Embedding). Let \( (\sigma, \mu, \nu) \) be such that Assumption 1 is fulfilled and that \( [X]_\infty = \infty \) a.s. Then \( |R_{reg}(\sigma, \mu, \nu)| = 1 \). Moreover,

\[
\mathbb{E}[[X]_{\tau_R}] = \int_\mathbb{R} x \mathbb{P}(dx) < \infty \quad \text{and} \quad \mathbb{E}[\tau_R] < \infty \quad \forall R \in R(\sigma, \mu, \nu).
\]

**Remark 5.** One can also apply the results of Rost and Cox–Wang [11, 34] to show that \( \tau_R \) minimizes \( \mathbb{E}[f(\tau)] \) among all \( \tau \in \mathcal{S}(\sigma, \mu, \nu) \) where \( f \) can be a convex, increasing function with \( f(0) = 0 \) which further fulfills some properties which depend in general on \( \sigma \).

2.3.2. Existence and Uniqueness for geometric Brownian motion \( (\sigma(t, x) = x) \). The case when \( X \) is a Geometric Brownian motion (GBM henceforth), i.e. \( \sigma(t, x) = x \), is not covered by Theorem 3 since although \( \mathbb{E}[[X]_\infty] = \infty \) we have \( [X]_\infty < \infty \) a.s. (otherwise one could write \( X \) as a time-changed Brownian motion via Dambis/Dubins–Schwarz). However, this case is especially important for the applications in finance (see Section 6).

**Assumption 2.** \( (\mu, \nu) \in M^2, \mu \leq_{cr} \nu \) and there exists an \( \epsilon > 0 \) such that

\[\text{supp}((\mu) \subset (0, +\infty) \quad \text{and} \quad \text{supp}((\nu) \subset (\epsilon, +\infty).\]

The assumption that \( \epsilon \) is strictly greater than 0 is needed for our arguments (see Remark 17 in the appendix) and does not cover the case \( \nu([0, \infty)) = 1, \nu(\{0\}) = 0 \). (However, note that one cannot hope for Root embeddings where the target measure has an atom at 0). We now state the embedding result for this situation (again, the proof is given in the appendix).

**Theorem 4** (Root Embedding via Geometric Brownian motion). Let \( \sigma(x) = x \) and Assumption 2 hold. Then \( |R_{reg}(\sigma, \mu, \nu)| = 1 \). Moreover,

\[
\mathbb{E}[[X]_{\tau_R}] = \int_\mathbb{R} x^2 \mu(dx) < \infty, \quad [\ln X]_{\tau_R} = \tau_R \quad \forall R \in R(\sigma, \mu, \nu)
\]

Further, for every other stopping time \( \tau \in \mathcal{S}(\sigma, \mu, \nu), \)

\[
\mathbb{E}[f(\tau_R)] \leq \mathbb{E}[f(\tau)]
\]

for every function \( f : [0, \infty) \to [0, \infty) \) which is convex, \( f(0) = 0 \) and \( f \) has a bounded right-derivative.

3. BACKGROUND ON VISCOSITY SOLUTIONS AND THE OBSTACLE PROBLEM

We briefly recall the definition and some basic stability properties of viscosity solutions in the parabolic setting (for a detailed exposition we refer to the User’s guide [12]) with some additional lemmas which will become useful in the subsequent sections. We then discuss the connection to reflected FBSDEs.

3.1. Sub- and supersolutions, properness and semirelaxed limits.

**Definition 11.** Let \( \mathcal{O} \) be a locally compact subset of \( \mathbb{R} \) and denote \( \mathcal{O}_T = (0, T) \times \mathcal{O} \) for \( T \in (0, \infty] \). Consider a function \( u : \mathcal{O}_T \to \mathbb{R} \) and define for \( (s, z) \in \mathcal{O}_T \) the parabolic superjet \( \mathcal{P}^{2+}_\mathcal{O} u(s, z) \) as the set of triples \((a, p, m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) which fulfill

\[
u(t, x) \leq u(s, z) + a(t - s) + \langle p, x - z \rangle + \frac{1}{2} \langle m(x - z), x - z \rangle + \alpha \left(|t - s| + |x - z|^2\right) \quad \text{as} \quad \mathcal{O}_T \ni (t, x) \to (s, z)
\]

Similarly we define the parabolic subjet \( \mathcal{P}^{2-}_\mathcal{O} u(s, z) \) such that \( \mathcal{P}^{2-}_\mathcal{O} u = -\mathcal{P}^{2+}(-u) \).

Example 4. If a function \( u \in C^{1,2}(\mathcal{O}_T, \mathbb{R}) \) then
\[
\mathcal{P}^{2,-}(t,x) = \left\{ \left( \frac{\partial u}{\partial t}(t,x), \frac{\partial u}{\partial x}(t,x), m \right) : m \leq \frac{\partial u}{\partial x^2}(t,x) \right\}
\]
\[
\mathcal{P}^{2,+}(t,x) = \left\{ \left( \frac{\partial u}{\partial t}(t,x), \frac{\partial u}{\partial x}(t,x), m \right) : m \geq \frac{\partial u}{\partial x^2}(t,x) \right\}
\]
hence
\[
\mathcal{P}^{2,-}(t,x) \cap \mathcal{P}^{2,+}(t,x) = \left\{ \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x^2} \right) \left| (t,x) \right. \right\}.
\]
(If \( u \) is only differentiable from the left or right, a similar statement holds and is important for the proof of our main result, see Lemma 5).

Remark 6. An equivalent definition is to consider the extrema of \( u - \varphi \) when \( \varphi \) is taken from an appropriate set of smooth test functions, see [12].

Definition 12. A function \( F : \mathcal{O}_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is proper if \( \forall (t,x,a,p) \in \mathcal{O}_T \times \mathbb{R} \times \mathbb{R} \)
\[
F(t,x,a,p,m) \leq F(t,x,b,p,m') \quad \forall m \geq m', \quad s \geq r.
\]
Denote the real-valued, upper semicontinuous functions on \( \mathcal{O}_T \) with \( USC(\mathcal{O}_T) \) and the lower semicontinuous functions with \( LSC(\mathcal{O}_T) \). A subsolution of the (forward problem)
\[
\left\{ \begin{array}{ll}
F(t,x,u,\partial_t u, Du, D^2 u) & = 0 \\
u(0,.) & = u_0(.)
\end{array} \right.
\]
is a function \( u \in USC(\mathcal{O}_T) \) such that
\[
F(t,x,a,p,m) \leq 0 \quad \forall (t,x) \in \mathcal{O}_T \quad \text{and} \quad (a,p,m) \in \mathcal{P}^{2,+}_O(u,t,x)
\]
\[
u(0,.) \leq u_0(.) \quad \text{on} \quad O.
\]
The definition of a supersolution follows by replacing \( USC(\mathcal{O}_T) \) by \( LSC(\mathcal{O}_T) \), \( \mathcal{P}^{2,+}_O \) by \( \mathcal{P}^{2,-}_O \) and \( \leq \) by \( \geq \). If \( u \) is a supersolution of (8) then we also say that \( F(t,x,u,\partial_t u, Du, D^2 u) \geq 0, \quad (a,p,m) \geq u_0(x) \) holds in viscosity sense (similar for subsolutions). Similarly we call a function \( v \) a viscosity \( \text{(sub-\&\-super\-)} \) solution of the backward problem
\[
\left\{ \begin{array}{ll}
G(t,x,v,\partial_t v, Dv, D^2 v) & = 0 \\
v(T,.) & = v_T(.)
\end{array} \right.
\]
if
\[
G(t,x,v,a,p,m) \leq 0 \quad \forall (t,x) \in \mathcal{O}_T \quad \text{and} \quad (a,p,m) \in \mathcal{P}^{2,+}_O(v,t,x)
\]
\[
v(0,.) \leq v_T(.) \quad \text{on} \quad O.
\]
The next lemma shows that finding a viscosity solution to (9) is equivalent to finding one to (8).

Lemma 4. Let \( T < \infty \). Then \( u \) is a viscosity solution of the forward problem (8) iff \( v(t,x) := u(T-t,x) \) is a viscosity solution of the backward problem (9) with \( v_T(.) = u_0(.) \) and
\[
G(t,x,v,a,p,m) = F(T-t,x,v,-a,p,m).
\]

Proof. Assume \( u \) is a viscosity solution. Let \( \varphi \in C^\infty(\mathcal{O}_T, \mathbb{R}) \) and assume \( (t,x) \mapsto v(t,x) - \varphi(t,x) \) attains a local maximum at \( (t,\hat{x}) \in \mathcal{O}_T \). Then \( (T-t,\hat{x}) \) is a local maximum of \( (t,x) \mapsto u(t,x) - \hat{\varphi}(t,x) \) with \( \hat{\varphi}(t,x) = \varphi(T-t,x) \), hence
\[
F(\cdot,\cdot,u,\partial_t \hat{\varphi}, D\hat{\varphi}, D^2 \hat{\varphi}) \bigg|_{(T-t,\hat{x})} = G(\cdot,\cdot,v,\partial_t \varphi, D\varphi, D^2 \varphi) \bigg|_{(t,\hat{x})}
\]
which is by Remark 6 sufficient to see that \( v \) is a viscosity solution of (9). The other implication follows similarly.

A strong feature of viscosity solutions, which is also the key for the proof of our main theorem, is that they are very robust under perturbations. The type of reasoning presented below is classic and called the Barles–Perthame method of semi-relaxed limits.

Definition 13. Let \( A \subseteq \mathbb{R}^m \) and \( (g_n)_n \) a sequence of functions \( g_n : A \to \mathbb{R} \) which is locally uniformly bounded. We define \( \gamma, \bar{g} : A \to \mathbb{R} \) as
\[
\gamma(p) = \limsup_{q \to p, n \to \infty} g_n(q).
\]
\[
\bar{g}(p) = \liminf_{q \to p, n \to \infty} g_n(q).
\]
Proposition 4. Let \((u^n)_n \subset USC(\mathcal{O}_T), \mathcal{O}\) a locally compact subset of \(\mathbb{R}\), \((F^n)_n\) a sequence of functions
\[
F^n : \mathcal{O}_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}
\]
such that each \(u^n\) is a subsolution of
\[
F^n (t, x, v, \partial_t v, D^2 v) = 0.
\]
Further, assume \((u^n)_n\) and \((F^n)_n\) are locally uniformly bounded. Then \(u\) is a subsolution of
\[
F (t, x, v, \partial_t v, D^2 v) = 0 \text{ on } \mathcal{O}.
\]
The analogous statement holds for a sequence of LSC(\(\mathcal{O}_T\)) functions which are supersolutions. Further, if \(\overline{u} = u\) then the convergence of \((u_n)\) to \(\overline{u} = u\) is locally uniform.

Proof. This is simply a restatement for parabolic PDEs of Proposition 4.3, Lemma 6.1 found in the User’s guide [12] combined with their subsequent remarks: the definition of \(\overline{u}\) as \(\limsup_{n \to \infty, (s, y) \to (t, x)} u^n (s, y)\) guarantees the existence of sequences
\[
n_j \to \infty, (t_j, x_j) \in \mathcal{O}_T \text{ s.t. } (t_j, x_j) \to (t, x) \text{ and } u_{n_j} (t_j, x_j) \to \overline{u}
\]
and it is clear that for every sequence \(\mathcal{O}_T \ni (t_j, x_j)\) converging to \((t, x)\) one has \(\limsup_j u_{n_j} (t_j, x_j) \leq \overline{u}\). This allows to apply [12, proposition 4.3] which implies that \(\forall (a, p, X) \in \mathcal{P}^{2+} (\mathcal{O}_T)\) there exists a sequence
\[
(t_j, x_j) \in \mathcal{O}_T, (p_j, m_j) \in \mathcal{P}^{2+} \text{ with } (t_j, x_j, u_{n_j} (t_j, x_j), a_j, m_j) \to (t, x, u (x), a, m) \in \mathcal{P}^{2+}_{\mathcal{O}_T} (t, x).
\]
However, each \(u_{n_j}\) is a subsolution so that for every \(j\) it holds that
\[
F_{n_j} (t_j, x_j, a_j, p_j, m_j) \leq 0
\]
and hence from the definition of \(F\) it follows that
\[
F (t, x, a, p, m) \leq 0.
\]
The proof for subsolutions follows the same argument. Finally, if \(\overline{u} = u\) then \(u\) must be continuous and the convergence \(u_n \to u\) must be uniform (as seen by [12, Remark 6.4]). □

3.2. Reflected FBSDEs and viscosity solutions of obstacle problems. Our goal is to show that knowing the Root barrier is equivalent to knowing the viscosity solution of a certain obstacle PDE, that is we are interested in the case \(F (t, x, r, a, p, m) = \min \left( r - h (x), a - \frac{\sigma^2 (t, x)}{2} m \right) \) where the initial condition \(u_0\) and the barrier \(h\) are given as the potential function of the initial and target measures and \(\sigma\) is continuous (we discuss the motivation for looking for such a PDE solution at the beginning of Section 4).

Definition 14. Let \(T \in [0, \infty), u_0 : \mathcal{O} \to \mathbb{R}\) and \(h, s \in [0, T) \times \mathbb{R}\). Denote with \(V_T (s, u_0, h)\) the (possibly empty) set of viscosity solutions
\[
\{ \min \left( u (t, x) - h (x), \left( \partial_t - \frac{x^2}{2} \Delta \right) u (t, x) \right) = 0, (t, x) \in (0, T) \times \mathbb{R}\}
\]
which are of linear growth uniformly in \(t\), i.e. \(\forall (t, x) \in [0, T) \times \mathbb{R}, \sup_{(t,x)\in[0,T)\times\mathbb{R}} \frac{|u(t,x)|}{1+|x|} < \infty\). We refer to this PDE as the obstacle problem on \((0, T) \times \mathbb{R}\) with barrier \(h\).

Remark 7. \(F (t, x, r, a, p, m) = \min \left( r - h (x), a - \frac{\sigma^2 (t, x)}{2} m \right) \) is proper and moreover the barrier \(h\) introduces an asymmetry in the definition of sub- and supersolutions: in this case Definition 12 says that \(u\) is a subsolution of (10) if \(u (0, \cdot) \leq u_0 (\cdot)\) and for every \((t, x) \in (0, T) \times \mathbb{R}\) such that \(u (t, x) > h (x)\) we have
\[
a - \frac{\sigma^2 (t, x)}{2} m \leq 0 \forall (a, p, m) \in \mathcal{P}^{2+}_{\mathcal{O}_T} u (t, x).
\]
Whereas \(u\) is a supersolution if \(u (0, \cdot) \geq u_0 (\cdot)\) and \(\forall (t, x) \in (0, T) \times \mathbb{R}\)
\[
a - \frac{\sigma^2 (t, x)}{2} m \geq 0 \forall (a, p, m) \in \mathcal{P}^{2-}_{\mathcal{O}_T} u (t, x) \text{ and } u (t, x) \geq h (x).
\]

We now discuss the abstract problem of existence and uniqueness of a viscosity solution to (2). First we recall the definition of a subclass of reflected forward-backward stochastic equation (RFBSDE) ([15, Section 8] applied with \(f = 0\)). Therefore fix \(T \in [0, \infty)\) and for each \(t \in [0, T]\) denote with \(\mathcal{F}_t\) the natural filtration of the Brownian motion \(\{W_s - W_t, t \leq s \leq T\}\), augmented by the \(\mathcal{P}\)-null sets of \(\mathcal{F}\). Take \((t, x) \in [0, T) \times \mathbb{R}\) and define for \(s \in [t, T]\) the forward-backward dynamics whose solution is the \(\mathcal{F}\)-adapted quadruplet \((X, Y, Z, K)\)
\[
\begin{align*}
X_s^{t,x} &= x + \int_t^s \sigma (T - r, X_r^{t,x}) dW_r, \\
Y_s^{t,x} &= u_0 (X_s^{t,x}) + K_s^{t,x} - K_t^{t,x} - \int_t^s Z_s^{t,x} dW_r, \\
Y_s^{t,x} &\geq h (X_s^{t,x}), t \leq s \leq T \text{ and } \int_t^s (Y_s^{t,x} - h (X_s^{t,x})) dK_s^{t,x} = 0,
\end{align*}
\]
where \((K_{s}^{t,x})_{s\in[0,T]}\) is an increasing and continuous process verifying \(K_{s}^{t,x} = 0\). In the above equation the meaning of the processes \(X\) and \(Y\) is clear and a rough interpretation of the processes \(Z\) and \(K\) is a follows: \(Z\) guides the evolution of \(Y\) via the Itô integral so that \(Y\) can hit the random variable \(u_{0}(X_{T})\) at horizon time \(T\). Note that both processes \(Y, Z\) are \((\mathcal{F}_{t})\)-adapted and so they do not “see the future”, nonetheless, \(u_{0}(X_{T})\) is attained at time \(t = T\). The process \(K\) ensures that \(Y\) does not go below the barrier \(h\): \(K\) pushes \(Y\) upwards whenever \(Y\) touches and tries to go below the barrier \(h\), else it remains inactive (that is constant).

**Proposition 5** (Stochastic representation for the obstacle PDE). Let \(u_{0}, h \in C(\mathbb{R}, \mathbb{R}), \sigma \in C([0, \infty) \times \mathbb{R}, \mathbb{R}), h(.) \leq u_{0}(.)\) and assume that for every \(t \in [0, \infty)\), \(\sigma(t,.)\), \(h(.)\), \(u_{0}(.)\) are Lipschitz continuous \((\sigma\ uniformly\ in\ t)\). Moreover, assume that

\[
\sup_{t \in [0, \infty)} \sup_{x \in \mathbb{R}} \left( \frac{|u_{0}(x)| + |\sigma(t,x)| + |h(x)|}{1 + |x|} \right) < \infty.
\]

Then there exists a \(v_{\infty} \in C(\{0, \infty\} \times \mathbb{R}, \mathbb{R})\) which is a viscosity solution of

\[
\min \left( v(t, x) - h(x), \left( \partial_{t} - \frac{\sigma^{2}}{2} \Delta \right) v(t, x) \right) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}
\]

Moreover, for every \(T \in (0, \infty)\)

1. there exists \(\forall (t, x) \in [0, T] \times \mathbb{R}\) a unique quadruple \((X_{s}^{t,x}, Y_{s}^{t,x}, Z_{s}^{t,x}, K_{s}^{t,x})_{s\in[0,T]}\) of \((\mathcal{F}_{s})\)-progressively measurable processes which satisfies for some \(c_{T} > 0\)

\[
\sup_{s \in [0, T]} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( \left| X_{t}^{s,x} \right|^{2} + \left| Y_{t}^{s,x} \right|^{2} \right) + \int_{t}^{T} \left| Z_{t}^{s,x} \right|^{2} dt + \left| K_{t}^{s,x} \right| \right] \leq c_{T} (1 + |x|)
\]

and this quadruple fulfills (13). Moreover, \(\forall (s, x) \in [t, T] \times \mathbb{R}\) the mapping \((s, x) \mapsto (X_{s}^{t,x}, Y_{s}^{t,x})\) is \(\mathbb{P}\)-a.s. jointly continuous and in particular \((X_{t}^{t,x}, Y_{t}^{t,x}, Z_{t}^{t,x}, K_{t}^{t,x}) \in \mathbb{R}^{4}\) (i.e. deterministic),

2. \((t, x) \mapsto Y_{t}^{T-t,x}\) is the unique element of \(\mathcal{V}_{T}(\sigma, u_{0}, h)\),

3. \(v_{\infty}|_{[0,T]} \in \mathcal{V}(\sigma, u_{0}, h)\).

**Proof.** Existence & uniqueness in \([0, T) \times \mathbb{R}, T < \infty\): Time reversal of (2) (see (9) and Lemma 4) shows that it is enough to deal with

\[
\min \left( v(t, x) - h(x), \left( -\partial_{t} - \frac{\sigma^{2}}{2} \Delta \right) v(t, x) \right) = 0, \quad (t, x) \in \mathcal{O}_{T} = (0, T) \times \mathbb{R}
\]

where \(\mathcal{O}(t, x) = \sigma(T - t, x)\). Continuity, existence and uniqueness of the viscosity solution \(v\) follows from Lemma 8.4, Theorems 8.5 and 8.6 in [15] respectively. The linear growth of \(u\) in its spatial variable follows from standard manipulations for RBSDEs. [15, Proposition 3.5] applied to the RFBSDE setting above (i.e. (13) with \(\sigma\ replaced by \(\sigma\)) yields the existence of a constant \(K_{T} > 0\) such that \(\forall (t, x) \in [0, T) \times \mathbb{R}\)

\[
\left| Y_{t}^{s,x} \right|^{2} \leq \mathbb{E} \left[ \sup_{s \in [t, T]} \left| Y_{s}^{t,x} \right|^{2} \right] \leq K_{T} \left( \mathbb{E} \left[ \left| u_{0}(X_{T}^{t,x}) \right|^{2} \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| h^{+}(s, X_{r}^{t,x}) \right| \right] \right) \leq c_{T} (1 + |x|^{2})
\]

with \(h^{+} := \max(0, h)\) and where the last inequality follows from the linear growth assumptions on \(u_{0}\) and \(h\) along with standard SDE estimates: \(\sup_{t \in [0, T]} \mathbb{E} \left[ \left| X_{T}^{t,x} \right|^{2} \right] \leq c_{T} (1 + |x|^{2})\) (see e.g. equation (4.6) of [16]). The solution to (2) follows via our Lemma 4.

Existence & uniqueness in \([0, \infty) \times \mathbb{R}\): Above gives a unique solution for every finite \(T > 0\). First note that for \(T, T' > 0\), \(Y_{T-t,x}^{T-t,x}\) and \(Y_{T'-t,x}^{T'-t,x}\) coincide on \([0, T \wedge T') \times \mathbb{R}\). By the comparison result in the appendix, Theorem 9, we can define \(v_{\infty}(t, x)\) as \(Y_{T-t,x}^{T-t,x}\) where \(T\) can be arbitrary chosen subject to \(T > t\). Since the definition of a viscosity solution just relies on local properties, \(v_{\infty}\) is a viscosity solution of (14).

**Remark 8.** In the above link between the solutions of (13) and (the time reversed version) of (10), (14) one can in general not expect that the solutions of (14) exist in a classical sense. The following formal argument gives at least an intuition why RFBSDE and obstacle PDEs are in a similar relation as SDEs and linear PDEs: suppose a sufficiently regular solution \(v\) of (14) exists. Via Itô’s formula it follows that:

\[
Y_{s}^{t,x} = v(s, X_{s}^{t,x}), \quad Z_{s}^{t,x} = (\sigma \nabla_{x} v)(s, X_{s}^{t,x}), \quad \text{and} \quad K_{s}^{t,x} = \int_{t}^{s} \left( -\partial_{t}v - \frac{\sigma^{2}}{2} \Delta v \right) (r, X_{r}^{t,x}) dr.
\]

The last condition in (13) then reads as

\[
\int_{t}^{T} (Y_{r}^{t,x} - h(X_{r}^{t,x})) dK_{r}^{t,x} = 0 \iff \int_{t}^{T} \left( (v - h) \left( -\partial_{t}v - \frac{\sigma^{2}}{2} \Delta v \right) \right) (r, X_{r}^{t,x}) dr = 0
\]

and the rhs explains the form of the PDE fulfilled by \(v\).
Remark 9. Proposition 5 applies as well to moving barriers (i.e. \( h \) could also depend on time) however we only need barriers constant in time. Note also that the above shows \( \{ v_\infty |_{[0,T] \times \mathbb{R}} \} = \mathcal{V}_T (\sigma, u_\mu, u_\nu) \) but it does not show that \( \{ v_\infty \} = \mathcal{V}_\infty (\sigma, u_\mu, u_\nu) \), viz. we only claim that \( v_\infty \) is a viscosity solution of the PDE (14) on the unbounded time domain but not that \( v_\infty \) is of linear growth uniformly in time (however, we this will be a simple consequence of Theorem 5).

The existence proof of (13) via penalization in [15] gives a lower bound for the first time the solution touches the barrier. This is intuitive clear but since it is useful for numerics we give the full proof.

Corollary 1. Let \( v_\infty \) be the solution of (14) as given by Theorem 5 and let \( w^0 \) to be the unique viscosity solution\(^4\) of

\[
\begin{cases}
\left( \partial_t - \frac{\sigma^2}{2} \Delta \right) w^0 (t, x) &= 0, \text{ on } (0, T) \times \mathbb{R} \\
 w^0 (0, \cdot) &= u_0 (\cdot)
\end{cases}
\]

Then

\[
w^0 (t, x) \leq v_\infty (t, x) \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\]

In other words, \( v_\infty \) hits the barrier \( h \) at time \( t \) only after \( w^0 \) has hit it at some time \( 0 \leq s \leq t \). Furthermore, if \( \sigma \) is uniformly parabolic and bounded then \( w^0 \in C^{1,2}((0,T) \times \mathbb{R}, \mathbb{R}) \) and can be written as a fundamental solution.

Proof. The statement follows by noting that \( v_\infty \) is a viscosity solution of (14), hence a supersolution of the heat equation (15). Applying a standard comparison result immediately shows \( w^0 \leq v_\infty \). Another way to see this without resorting to comparison results and which gives another construction of a solutions to the obstacle problem is the following: the existence proof for (13) was done in [15] via penalization arguments. For \( n \in \mathbb{N} \) consider the penalized obstacle problem

\[
\partial_t w^n (t, x) - \frac{1}{2} \sigma^2 (t, x) \Delta w^n (t, x) = n (w^n (t, x) - h (x) ), \quad (t, x) \in \mathcal{O}_T,
\]

\[
w^n (0, x) = u_0 (x), \quad x \in \mathbb{R},
\]

where \( a^- := \min \{ a, 0 \} \) for any \( a \in \mathbb{R} \) implying \( (w^n (t, x) - h (x) )^- := \min \{ w^n (t, x) - h (x), 0 \} \). Intuitively, the solution of the penalized equation approximates that of (2) as \( n \to \infty \) since the term \( n (w^n - h)^- \) blows up for the points which do not satisfy the barrier condition and so in order for a well behaved solution to exist in the limit the barrier condition has to be verified. To shorten the discussion, along the proof it is shown the sequence \( (w^n)_{n \in \mathbb{N}} \) converges and that \( w^n \uparrow u \) (as \( n \to \infty \)), i.e. the sequence is monotonic increasing (see [15, Theorem 8.5]). It follows that \( w^0 (t, x) \leq u (t, x) \) for all \( (t, x) \in [0, T] \times \mathbb{R} \) with \( w^0 \) the solution to the penalized PDE for \( n = 0 \), equation (15).

Standard results (see for example [18, Chapter 1, Section 7, Theorem 12]) guarantee that under the stronger assumption of boundedness and uniform parabolicity of \( \sigma \) there exists a smooth solution and since every smooth solution is a viscosity solution the statement follows.

4. Root’s barrier and the parabolic obstacle problem

We are now in position to prove the main results: a one–to–one correspondence between regular Root barriers and viscosity solutions.

4.1. From the Root barrier to the obstacle problem. As pointed out in Section 2, potential functions are a powerful way to keep track of the evolution of the distribution of a local martingale. Let \( R \in \mathcal{R} (\sigma, \mu, \nu) \) and consider the potential function of the stopped process \( X^{\tau_n} \), namely \( u (t, x) = u_{X^{\tau_n}} (t, x) \equiv \mathbb{E} [ | X^{\tau_n}_t - x | ] \). Tanaka’s formula immediately gives

\[
u (t, x) = u_\mu (x) - \mathbb{E} [ L^{X^{\tau_n}}_t ],
\]

where \( (L^X_t)_{t \geq 0} \) denotes the local time of \( (X_t)_{t \geq 0} \), i.e. we see for every fixed \( x \in \mathbb{R} \) that \( t \mapsto u (t, x) \) decreases until we have subtracted enough such that it matches \( u_\mu (x) \). While one cannot hope that \( u \) is a classical PDE solution (due to the kinks that appear when \( u \) touches the barrier \( u_\mu \)) we give a formal derivation for the PDE satisfied by \( u \) which then motivates our proof of Theorem 5. For brevity also assume \( \sigma \equiv 1 \) (i.e. we embed into Brownian motion). Consider a sequence \( (f^n)_n \subset C^\infty (\mathbb{R}, \mathbb{R}) \) such that \( f^n (\cdot) \to_{n \to \infty} \cdot \mid \) uniformly, hence \( \nabla f^n (\cdot) \to_{n \to \infty} \cdot \text{sgn} (\cdot) \) and \( \Delta f^n (\cdot) \to_{n \to \infty} \cdot \delta_0 (\cdot) \). Set

\[
\nu^n (t, x) := - \mathbb{E} [ f^n (X^{\tau_n}_t - x) ] .
\]

\(^4\)This equation is a standard parabolic PDE whose unique solution is known to exist under the assumptions of Theorem 5, see e.g. the proof of Theorem 8.5 in [15] or think of the RBSDE (13) when \( h = - \infty \) (implying that \( K_t = 0 \)).
and since \( f^n \) is smooth, \( \Delta u^n (t, x) = -\mathbb{E} [\Delta f^n (X_t^n - x)] \) and from Itô’s formula applied to the right hand side of (16) it follows that
\[
(17) \quad u^n (t, x) = - \int_\mathbb{R} f^n (y - x) \mu (dy) - \frac{1}{2} \int_0^t \mathbb{E} [\Delta f^n (X_r - x)] \, dr.
\]
Hence \( \partial_t u^n (t, x) = - \frac{1}{2} \mathbb{E} [\Delta f^n (X_r - x)] \) and putting the above together shows
\[
\partial_t u^n - \frac{1}{2} \Delta u^n = \frac{1}{2} \mathbb{E} [\Delta f^n (X_t^n - x) - \Delta f^n (X_r - x)] \leq 0
\]
that is \( u \) is a supersolution of the heat equation. Further, equation (17) applied with \( t = \infty \) and using that \( X_{\tau_n} \sim \nu \) also gives
\[
u^n (t, x) - u^n (x) = \frac{1}{2} \int_\mathbb{R} \mathbb{E} [\Delta f^n (X_r - x)] \, dr.
\]
Now for \( (t, x) \in R \), also \( (s, x) \in R \) for \( s \geq t \) and one expects that \( \Delta f^n (X_s^n - x) \approx 0 \) a.s. since for \( n \) big enough
\[
\text{supp} (\Delta f^n (\cdot - x)) \approx \{ x \},
\]
hence the Itô correction vanishes and above reads in the limit as \( n \to \infty \) as
\[
u = u \nu \text{ on } R.
\]
Now if \( (t, x) \in R^c \) and \( t \geq \tau_n \) then the properties of the Root barrier imply that \( X_{\tau_n} \neq x \), hence \( \Delta f^n (X_{\tau_n} - x) \approx 0 \) for \( n \) big enough. We therefore arrive (as \( n \to \infty \)) at
\[
\partial_t u = \Delta u \quad \text{for } (t, x) \in R^c,
\]
\[
u (0, x) = u \mu (x).
\]
To conclude this (very formal) argument note that \( R \cup R^c = [0, \infty] \times \mathbb{R} \) and therefore putting all the above shows us that
\[
\min (u - u \nu, \partial_t u - \Delta u) = 0 \quad \text{on } (0, \infty) \times \mathbb{R},
\]
\[
u (0, \cdot) = u \mu (\cdot).
\]
The proof of Theorem 5 makes this type of reasoning precise via the method of semi-relaxed limits.

**Theorem 5.** Let \( (\sigma, \mu, \nu) \) be either as in Theorem 3 or as in Theorem 4. Let \( R \in \mathcal{R} (\sigma, \mu, \nu) \) and denote with \( u \) the potential function of the stopped process \( X_{\tau_n} = (X_{t \land \tau_n})_t \), i.e.
\[
u (t, x) = u_{X_{\tau_n}} (t, x) = -\mathbb{E} [\| X^n_{\tau_n} - x \|].
\]
Then \( \{ u \} = \mathcal{V}_\infty (\sigma, u \mu, u \nu), \) in other words, the potential function of \( X_{\tau_n} \) is the unique viscosity solution of linear growth of the obstacle problem (10) with initial data \( u \mu \), barrier \( u \nu \) and heat coefficient \( \sigma^2 \),
\[
\left\{ \begin{array}{ll}
\min (u - u \nu, \partial_t u - \frac{\sigma^2}{2} \Delta u) &= 0, \quad \text{on } (0, \infty) \times \mathbb{R} \\
u (0, \cdot) &= u \mu (\cdot).
\end{array} \right.
\]
Moreover,
1. for every \( x \in \mathbb{R} \) \( t \mapsto u (t, x) \) is non-increasing and \( u \nu (x) \leq u (t, x) \leq u \mu (x) \quad \forall \, (t, x) \),
2. \( u|_{R=\mathbb{R}} = u |_{R} \) and \( \forall (t, x) \lim_{t \to \infty} u (t, x) = u \nu (x) \),
3. \( u \) is Lipschitz in space (uniformly in time),
\[
\sup_{t \in [0, \infty)} \sup_{x \neq y} \frac{|u (t, x) - u (t, y)|}{|x - y|} < \infty.
\]

**Proof.** We have to show that \( u \) is a viscosity solution of (18) and to show uniqueness of the solutions via the comparison theorem given in the appendix we also need to establish that \( u \) is of linear growth. First note, that to verify that \( u \) is the viscosity of (18) it is enough to show that (in viscosity sense)
\[
\begin{align*}
\partial_t - \frac{\sigma^2}{2} \Delta u &= 0 \quad \text{on } [0, \infty) \times \mathbb{R}, \\
\partial_t - \frac{\sigma^2}{2} \Delta u &= 0 \quad \text{on } R^c,
\end{align*}
\]
The first inequality follows immediately via conditional Jensen
\[ u(t, x) = -\mathbb{E} \left[ |X_t^{nR} - x|^2 \right] \geq \mathbb{E} \left[ -|X_{\tau_R} - x| \right] \mathbb{E} \left[ \mathcal{F}_{t \wedge \tau_R} \right] = -\mathbb{E} \left[ |X_{\tau_R} - x| \right] = u_v(x). \]
For the other inequalities we approximate (18) and conclude by the Barles–Perthame method of semi-relaxed limits.

**Step 1.** \( u^n(t, x) := \mathbb{E} [\psi^n(X_t^{nR} - x)]. \) Define the sequence \((\psi_n) \subset C^2(\mathbb{R}, \mathbb{R})\) as
\[ \psi_n(x) = \int_0^x \int_0^y n\phi(nz) \, dz \, dy \, \forall x \in \mathbb{R} \]
where \( \phi \) is the usual Gaussian scaled to the unit disc
\[ \phi(x) = \begin{cases} \exp \left( -\frac{1}{t^2} \right) & \text{for } |x| < 1 \\ 0 & \text{otherwise.} \end{cases} \]
Especially note that \( \psi_n(\cdot) \to |\cdot| \) uniformly, \( \Delta \psi_n(\cdot) \) is continuous, \( 0 \leq \Delta \psi_n \leq O(n) \) and \( \text{supp} (\Delta \psi_n) \subset [-\frac{1}{n}, \frac{1}{n}] \) (we could replace \( \psi^n \) by any other sequence with this properties). Further introduce
\[ u^n(t, x) = -\mathbb{E} [\psi_n(X_t^{nR} - x)], \quad u^n_\nu(x) = -\int_\mathbb{R} \psi^n(y - x) \nu(dy), \quad u^n_\mu(x) = -\int_\mathbb{R} \psi^n(y - x) \mu(dy). \]
Since \( \psi_n(\cdot) \to |\cdot| \) uniformly we have \( \mathbb{P}\text{-a.s.} \)
\[ |\psi^n(X_t^{nR} - .) - |X_t^{nR} - .|_{\infty;[0, \infty) \times \mathbb{R}} = \sup_{(t,x) \in [0, \infty) \times \mathbb{R}} |\psi^n(X_t^{nR} - x) - |X_t^{nR} - x|| \to n \to \infty 0 \]
hence we get uniform convergence of \( u^n, u^n_\nu \) and \( u^n_\mu \), i.e.
\[ |u^n - u_{\infty;[0, \infty) \times \mathbb{R}}|_n \to 0, \quad |u^n_\nu - u_\nu|_{\infty, \mathbb{R}} \to 0, \quad |u^n_\mu - u_\mu|_{\infty, \mathbb{R}} \to 0. \]
Denote the complement of \( R \) with
\[ R^c = ([0, \infty) \times [-\infty, \infty]) \setminus R \]
and note that since \( R \) is closed, \( R^c \) is open. Further, by dominated convergence we have \( \forall x \in \mathbb{R}, \forall n \in \mathbb{N} \) that
\[ \lim_{t \to \infty} u(t, x) = u_\mu(x) \quad \text{and} \quad \lim_{t \to \infty} u^n(t, x) = u^n_\mu(x). \]

**Step 2.** \( u - u_\nu = 0 \) on \( R \). Fix \( x \in \mathbb{R} \) and apply the Itô-formula to \( -\psi^n(\cdot - x) \) and the local martingale \( X_t^{nR} \). After taking expectations and using Fubini’s theorem we arrive at
\[ u^n(t, x) = u^n_\mu(x) - \int_0^t \mathbb{E} \left[ \frac{\sigma^2(r, X_r)}{2} \Delta \psi^n(X_r - x) 1_{r < \tau_R} \right] dr. \]
Taking \( \lim_{t \to \infty} \) on both sides in the above implies by (22) that
\[ u^n_\mu(x) = u^n_\mu(x) - \int_0^\infty \mathbb{E} \left[ \frac{\sigma^2(r, X_r)}{2} \Delta \psi^n(X_r - x) 1_{r < \tau_R} \right] dr. \]
Since \( \frac{\sigma^2}{2} \Delta \psi^n \geq 0, (23) \) shows also that \( u^n \) is non-increasing in time and by subtracting (24) from (23) we arrive at
\[ u^n(t, x) - u^n_\mu(x) = \int_0^\infty \mathbb{E} \left[ \frac{\sigma^2(r, X_r)}{2} \Delta \psi^n(X_r - x) 1_{r < \tau_R} \right] dr \geq 0. \]
The uniform convergence given in (21) implies that \( t \mapsto u(t, x) \) is non-increasing, i.e. \( \forall (t, x) \in [0, \infty) \times \mathbb{R} \) we have that
\[ u(t + r, x) \leq u(t, x) \quad \forall r \geq 0. \]
Fix a point \((t, x) \in R^c \) (\( R^o \) denoting the interior of \( R \)). Because \( R \) is a Root barrier this implies that \( \forall r \geq t \) also \( (r, x) \in R^c \) and since \( R^c \) is open we conclude that \( \exists \eta_0 \) s.t.
\[ \{ (r, y) \in [0, \infty) \times [-\infty, \infty] : r \geq t, |x - y| < \frac{1}{\eta_0} \} \subset R^o. \]
Since \( \text{supp} (\Delta \psi^n) \subset [-n^{-1}, n^{-1}] \) this shows \( \Delta \psi^n(X_r - x) 1_{r < \tau_R} = 0 \) for all \( r \geq t \) and all \( n > \eta_0 \), hence
\[ \lim_{n \to \infty} (u^n - u_\mu^n)(t, x) = \int_0^\infty \mathbb{E} \left[ \frac{\sigma^2(r, X_r)}{2} \lim_{n \to \infty} \Delta \psi^n(X_r - x) 1_{r < \tau_R} \right] dr = 0 \quad \forall (t, x) \subset R^c. \]
By continuity of \( u \) and \( u_\nu \) we can conclude that
\[
\left. u \right|_{\partial R} = \left. u_\nu \right|_{\partial R}.
\]

**Step 3.** \( \left( \partial_t - \frac{\sigma^2}{2} \Delta \right) u \geq 0 \) on \([0, \infty) \times \mathbb{R} \). From (23) it follows that \( u^n(t, x) \) has a right- and left-derivative \( \forall (t, x) \in [0, \infty) \times \mathbb{R} \); to see this take
\[
\partial_{t+} u^n(t, x) = \lim_{\epsilon \downarrow 0} \epsilon^{-1} (u(t, x + \epsilon) - u(t, x))
\]
\[
= -\mathbb{E} \left[ \lim_{\epsilon \downarrow 0} \epsilon^{-1} \int_t^{t+\epsilon} \frac{\sigma^2(t, X_t)}{2} \Delta \psi^n(x_t - x) 1_{t < \tau_n} \, dt \right]
\]
\[
= -\mathbb{E} \left[ \frac{\sigma^2(t, X_t)}{2} \Delta \psi^n(x_t - x) 1_{t < \tau_n} \right]
\]
similarly it follows that the left derivative \( \partial_{t-} u^n \) is given as
\[
\partial_{t-} u^n(t, x) = -\mathbb{E} \left[ \frac{\sigma^2(t, X_t)}{2} \Delta \psi^n(x_t - x) \, 1_{t \leq \tau_n} \right].
\]

Note that for every \( t \in [0, \infty) \) \( \partial_{t-} u^n(t, .) \) \( \partial_{t+} u^n(t, .) \) \( C^\infty(\mathbb{R}, \mathbb{R}) \); further, since \( \frac{\sigma^2(t, X_t)}{2} \Delta \psi^n(x_t - x) \) is non-negative we conclude
\[
\partial_{t-} u^n(t, x) \leq \partial_{t+} u^n(t, x) \leq 0.
\]

(Note that for \( n < \infty \) we cannot expect the above inequality to be an equality; e.g. consider \( \nu = \delta_0, \mu = N(0, 1), \sigma \equiv 1 \) which is solved by \( R = \{ (t, x) : t \geq 1 \} \), hence \( \tau_n \equiv 1 \). From the definition of \( u^n \) it follows that we can exchange differentiation in space and expectation to arrive at
\[
\frac{\sigma^2(t, x)}{2} \Delta u^n(t, x) = -\frac{\sigma^2(t, x)}{2} \mathbb{E} [\Delta \psi^n(x_t^n - x)] \leq 0 \text{ on } [0, \infty) \times \mathbb{R},
\]
which is a continuous function in \((t, x)\). Lemma 5 shows that \( \forall (a, p, m) \in \mathcal{P}_2^n \) \( u^n(t, x) \)
\[
a \geq \partial_{t-} u^n(t, x) \text{ and } m \leq \Delta u^n(t, x).
\]
Hence, by (26) and (25)
\[
a - m \geq \partial_{t-} u^n(t, x) - \frac{\sigma^2(t, x)}{2} \Delta u^n(t, x)
\]
\[
= \frac{1}{2} \mathbb{E} \left[ \sigma^2(t, x) \Delta \psi^n(x_t^n - x) - \sigma^2(t, x) \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} \right].
\]

Splitting the term inside the expectation gives
\[
\partial_{t-} u^n(t, x) - m \geq \frac{1}{2} \mathbb{E} \left[ \sigma^2(t, x) \Delta \psi^n(x_t^n - x) - \sigma^2(t, x) \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} \right] =: I_n(t, x)
\]
\[
+ \frac{1}{2} \mathbb{E} \left[ \sigma^2(t, x) \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} - \sigma^2(t, x) \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} \right] =: I_n(t, x).
\]

We conclude that \( u^n \) is a supersolution of \( \left( \partial_t - \frac{\sigma^2}{2} \Delta \right) u - \frac{1}{2} I_n + I_n(t, x) = 0 \) on \((0, \infty) \times \mathbb{R}\). Further,
\[
I_n(t, x) = \mathbb{E} \left[ \sigma^2(t, x) \Delta \psi^n(x_t^n - x) - \sigma^2(t, x) \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} \right]
\]
\[
= \sigma^2(t, x) \mathbb{E} [\Delta \psi^n(x_t^n - x) 1_{t \leq \tau_n}] \geq 0
\]
hence \( u^n \) is also a viscosity supersolution of
\[
\left\{ \begin{array}{l}
\partial_t u - \frac{\sigma^2}{2} \Delta u - \frac{1}{2} I_n = 0 \text{ on } (0, \infty) \times \mathbb{R} \\
u(0, .) = u^n_\mu(\cdot),
\end{array} \right.
\]
Using the Lipschitz property of \( \sigma \), \( \text{supp}(\Delta \psi^n) = [-n^{-1}, n^{-1}] \) and that \( |\Delta \psi^n|_\infty \leq c n \) we estimate
\[
|I_n(t, x)| \leq \mathbb{E} \left[ |\sigma^2 (t, x) - \sigma^2 (t, X_t)| \Delta \psi^n(x_t - x) 1_{t \leq \tau_n} \right]
\]
\[
\leq \sigma_{Lip} \frac{2 \sigma_{LG}}{n} \left[ 1 + x + \frac{1}{n} \right] \mathbb{E} [\Delta \psi^n(x_t - x) 1_{t \leq \tau_n}]
\]
\[
\leq 4 \sigma_{Lip} \sigma_{LG} \left[ x + \frac{1}{n} \right] \mathbb{E} [\inf \{ |X_t - x| \leq n^{-1} \}]
\]
\[
= 4 \sigma_{Lip} \sigma_{LG} \left[ x + \frac{1}{n} \right] \mathbb{P} \left( |X_t - x| \leq n^{-1} \right)
\]
appears in the proof above. The lemma below is a generalization of this for functions which are only left- and right-differentiable which is enough to conclude that

\( \|X_t - x\| \leq n^{-1} \)

for every compact \( K \subset [0, \infty) \times \mathbb{R} \) we have

\[
\lim_{n \to \infty} \sup_{(t,x) \in K} \mathbb{P} \left[ \|X_t - x\| \leq n^{-1} \right] = 0
\]

since by Assumption 1 the process \( X \) has a continuous density with respect to the Lebesgue measure on \((\mathbb{R}, \mathcal{B})\) and

\[
\mathbb{P} \left[ |X_t - x| \leq n^{-1} \right] = \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} |y - x| f(t,y) \, dy 
\]

\[
\leq \frac{1}{n} \int_{\mathbb{R}} f(t,y) \, dy \to 0 \text{ as } n \to \infty
\]

uniformly in \((t,x)\), therefore \( I_n \to 0 \) locally uniformly on \([0, \infty) \times \mathbb{R}\). By step 1, \( u^n \to u \) and \( u^n(0,\cdot) \to u_\mu(\cdot) \) locally uniformly as \( n \to \infty \). The stability of viscosity solutions, Proposition 4, implies that \( u \) is a viscosity supersolution of

\[
(27) \quad \begin{cases} 
\left( \partial_t - \frac{\sigma^2}{2} \Delta \right) u = 0 \text{ on } [0, \infty) \times \mathbb{R} \\
u(0,\cdot) = u_\mu(\cdot).
\end{cases}
\]

which proves the desired inequality.

**Step 4.** \( \left( \partial_t - \frac{\sigma^2}{2} \Delta \right) u = 0 \) on \( \mathbb{R}^c \). From Lemma 5 it follows that if \( \partial_{-} u^n(t,x) < \partial_{+} u^n(t,x) \) then \( \mathcal{P}^{2,+} u^n(t,x) = \emptyset \) (in which case we are done) and if \( \partial_{-} u^n(t,x) = \partial_{-} u^n(t,x) = \partial_{+} u^n(t,x) \) then \( \forall (a,p,m) \in \mathcal{P}^{2,+} u^n(t,x) \), \( a = \partial_t u^n(t,x) \) and \( m \geq \Delta u^n(t,x) \). Hence, in the latter case we have \( \forall (a,p,m) \in \mathcal{P}^{2,+} u^n(t,x) \) that

\[
a - m \leq \partial_t u^n(t,x) - \frac{\sigma^2}{2} (t,x) \Delta u^n(t,x).
\]

Proceeding as in step 3 we see that \( u^n \) is a subsolution of

\[
(\partial_t - \frac{\sigma^2}{2} \Delta - \frac{1}{2} (I_n + I_{I_n}) \) \quad u = 0 \text{ on } (0, \infty) \times \mathbb{R}
\]

| \( u(0,\cdot) = u^n(\cdot) \). |

We already know that \( I_n \to 0 \) locally uniformly and now show that \( I^n \to 0 \) locally uniformly on \( \mathbb{R}^c \): since \( R \) is a Root barrier we have

\( (\tau_{R} + r, X_{\tau_{R}}) \in R \) \( \forall r \geq 0 \), hence if \( (t,x) \in \mathbb{R}^c \) and \( t \geq \tau_{R} \) then \( X_{\tau_{R}} \neq x \). Therefore

\[
\lim_{n \to \infty} \sup_{(t,x) \in K} \Delta \psi^n(X_{\tau_{R}} - x) \leq 0 \quad \text{for every compact } K \subset \mathbb{R}^c
\]

which is enough to conclude that \( I_n \) converges locally uniformly on \( \mathbb{R}^c \) to 0, i.e. for every compact \( K \subset \mathbb{R}^c \)

\[
\lim_{n \to \infty} \sup_{(t,x) \in K} I_n(t,x) \leq |\sigma^2|_{\infty,K} \lim_{n} \mathbb{E} \left[ \sup_{(t,x) \in K} \Delta \psi^n(X_{\tau_{R}} - x) \leq 0 \right] = 0.
\]

The stability results, Proposition 4, implies that \( u \) is a subsolution of

\[
(\partial_t - \frac{\sigma^2}{2} \Delta) u = 0 \text{ on } \mathbb{R}^c,
\]

| \( u(0,\cdot) = u_\mu(\cdot) \). |

Together with the result from step 3 this shows that \( u \) is a solution in viscosity sense.

**Step 5.** Putting the above together shows that \( u \) is a viscosity solution of the obstacle problem (18). To see that \( u \) is of linear growth, recall that by definition \( u(0,x) = u_\mu(x) \) and in step 2 we showed that \( \forall t \in \mathbb{R}, t \to u(t,x) \) is non-increasing. Further, we know \( u(t,x) \geq u_\nu(x) \forall (t,x) \) from the estimate (20), hence \( |u(t,x)| \leq |u_\mu(x)| + |u_\nu(x)| \). This together, with the properties of potential functions, Proposition 1 and Lemma 2, implies \( \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} \frac{u(t,x)}{1 + |t||x|} < \infty \). This shows that the assumptions of the comparison result, Theorem 9, are met and we conclude that \( \{u\} = \mathcal{V}_\infty(\sigma, u_\mu, u_\nu) \).

For a function \( v \in C^{1,2}([0, \infty) \times \mathbb{R}, \mathbb{R}) \) one has \( \mathcal{P}^{2,-} v(t,x) \cap \mathcal{P}^{2,+} v(t,x) = \left\{ \left( \frac{\partial v}{\partial t}(t,x), \frac{\partial v}{\partial x}(t,x), \frac{\sigma^2 v}{2}(t,x) \right) \right\} \). The lemma below is a generalization of this for functions which are only left- and right-differentiable which appear in the proof above.
Lemma 5. Let $v \in C((0, \infty) \times \mathbb{R}, \mathbb{R})$ and assume that $\forall (t, x) \in (0, \infty) \times \mathbb{R}$, $v$ has a left- and right-derivative, i.e. the following limits exist
\[
\partial_+ v(t, x) = \lim_{\epsilon \searrow 0} \frac{v(t + \epsilon, x) - v(t, x)}{\epsilon} \quad \text{and} \quad \partial_- v(t, x) = \lim_{\epsilon \nearrow 0} \frac{v(t + \epsilon, x) - v(t, x)}{\epsilon}.
\]
If $\partial_- v(t, x) \leq \partial_+ v(t, x)$ then
\[
a \in [\partial_- v(t, x), \partial_+ v(t, x)] \quad \forall (a, p, m) \in P^{2-} v(t, x).
\]
If $\partial_- v(t, x) < \partial_+ v(t, x)$ then $P^{2+} v(t, x) = \emptyset$ and if $\partial_- v(t, x) > \partial_+ v(t, x)$ then $\forall (a, p, m) \in P^{2+} v(t, x)$.

Proof. Every element $(a, p, m) \in P^{2-} v(t, x)$ satisfies
\[
v(t + \epsilon, x) - v(t, x) \geq a \epsilon + o(\epsilon) \quad \epsilon \to 0.
\]

}\[
\begin{align*}
\text{Theorem 5.} & \quad \text{Let } \sigma, \mu, \nu \text{ be either as in Theorem 3 or as in Theorem 4. Then there exists a unique viscosity solution of the obstacle problem (18), i.e. } \{ u \} = \mathcal{V}_{\infty}(\sigma, \mu, \nu), \text{ and} \\
\{ (t, x) \in [0, \infty) \times [-\infty, \infty) : u(t, x) = u_{\nu}(x) \} = R
\end{align*}
\]
where $\{ R \} = \mathcal{R}_{\text{reg}}(\sigma, \mu, \nu)$, i.e. $R$ is the unique $(\mu, \nu)$-regular Root barrier which solves the Skorohod embedding problem given by $(\sigma, \mu, \nu)$.

Proof. Above assumptions guarantee that $(\sigma, \mu, \nu)$ fulfills the conditions of Proposition 5, i.e. there exists a viscosity solution of the obstacle problem with initial data $u_{\nu}$, barrier $u_{\nu}$ and heat coefficient $\sigma^2$, i.e. $\mathcal{V}_{T}(\sigma, \mu, u_{\nu}) = \{ u_{[0,T] \times \mathbb{R}} \}$ for every $T \in (0, \infty)$. On the other hand Theorem 3 gives the existence of a unique $(\mu, \nu)$-regular Root barrier, i.e. $|\mathcal{R}_{\text{reg}}(\sigma, \mu, \nu)| = 1$ and from Theorem 5 we know that
\[
\begin{align*}
\{ u \} = \mathcal{V}_{\infty}(\sigma, \mu, u_{\nu}).
\end{align*}
\]

It remains to show (28). From step 2 in the proof of Theorem 5 it follows that
\[
R \subset \{ (t, x) \in [0, \infty) \times [-\infty, \infty) : u(t, x) = u_{\nu}(x) \} =: Q.
\]
To see the other inclusion $Q \subset R$ note that by definition of $u$ we can apply the Tanaka formula to see
\[
\begin{align*}
\begin{aligned}
u(t, x) &= u_{\mu}(x) - \mathbb{E}[L_{t}^x] \
\end{aligned}
\end{align*}
\]
where $L_{t}^x$ denotes the local time of the stopped process $X^\tau = (X_{\tau \land t})$. Hence
\[
\begin{align*}
\begin{aligned}
u(t + r, x) - 
u(t, x) &= \mathbb{E}[L_{t}^x - L_{t+r}^x] \
\end{aligned}
\end{align*}
\]
Now if $(t, x) \in Q$, then $(t + r, x) \in Q$ and so $\nu(t + r, x) - \nu(t, x) = u_{\nu} - u_{\nu}$ (this follows from properties of the PDE solution as stated in Theorem 5). Hence
\[
\begin{align*}
L_{t}^x = L_{t+r}^x \quad \forall (t, x) \in Q.
\end{align*}
\]
Therefore it is sufficient to show that if $(t, x) \in R^c$ then $\exists r > 0$ s.t.
\[
\mathbb{P}[L_{t+r}^x > L_{t}^x] > 0
\]
However, as a consequence of the Ray–Knight theorem [24, Chapter 22, Corollary 22.18] we have \( \forall (t, x), r \geq 0 \) that

\[
\{ L_{t+t+r}^\sigma - L_t^\sigma > 0 \} = \left\{ \inf_{s \in [t, t+r]} X_t^\sigma < x < \sup_{s \in [t, t+r]} X_t^\sigma \right\}
\]

so it only remains to to show that \( \forall (t, x) \in R^c \) there exists a \( r > 0 \) s.t.

\[
P \left[ \inf_{s \in [t, t+r]} X_t^\sigma < x < \sup_{s \in [t, t+r]} X_t^\sigma \right] > 0.
\]

If the assumptions of Theorem 3 hold, i.e., \( X \) a geometric BM this follows immediately and if we use the assumptions of Theorem 3 we can use Dambins/Dubins–Schwarz to see that \( X = X_0 + W_{[X]} \) for some BM \( W \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \), and since \( \mathbb{P}[\tau_R > 0] > 0 \) we have

\[
P \left[ \inf_{s \in [t, t+r]} X_0 + W_{[X]} \wedge \tau_R < x < \sup_{s \in [t, t+r]} X_0 + W_{[X]} \wedge \tau_R \right] > 0.
\]

\[\Box\]

Remark 11. Let \( Q \in R(\sigma, \mu, \nu) \). Applying Theorem 5 with \( Q \) leads to a function \( u(t, x) := \mathbb{E}[X_t^\sigma - x] \), \( u \in \mathcal{V}(\sigma, u_\mu, u_\nu) \) and applying Theorem 6 to \( u \) yields the barrier \( Q \cup N^{\mu, \nu} \). In other words, the PDE representation always chooses the unique \((\mu, \nu)\)-regular barrier.

Remark 12. Consider \((\sigma, \mu, \nu) := (1 \leq 1, \delta_0, \mathcal{N}(0, 1))\) and note that \( \sigma(t, x) = 1 \leq 1 \) does not fall into our setting. Obviously any fixed (deterministic) \( \tau \geq 1 \) is an element of \( S(\sigma, \mu, \nu) \) but only \( \tau = 1 \) is the minimal solution to the embedding problem (in the sense of Definition 6). Note that the above construction via the obstacle PDE chooses the minimal solution \( \tau = 1 \).

4.3. The reversed Root barrier (Rost’s barrier). Root’s barrier \( R \) resp. the associated hitting time lets the process \( X \) diffuse as much as possible before it hits the barrier. Motivated by this, one can ask for the other extreme: how to build a barrier \( R \) such that its hitting time solves the Skorokhod embedding problem and stops the process \( X \) as soon as possible. Indeed such a construction was given by Rost [32] who was led to this topic by the Chacon–Ornstein ergodic theorem and an extension of the filling scheme from discrete to continuous time. A closed subset \( \bar{R} \) of \([0, \infty) \times [-\infty, \infty] \) is a Root barrier if

1. \((t, x) \in \bar{R} \) implies \((s, x) \in \bar{R} \) \( \forall s \leq t \),
2. \((0, x) \in \bar{R} \) \( \forall x \in [-\infty, \infty] \),
3. \((t, \pm \infty) \in \bar{R} \) \( \forall t \in [0, +\infty] \).

Following the very same approach as in Theorem 5 via semirelaxed limits allows one to show that if \( \bar{R} \) is a Root barrier which solves the Skorokhod embedding problem for \((\sigma, \mu, \nu)\) then the continuous function

\[
u(t, x) := u(t, x) := u_{\bar{R}}(t, x) - u_\nu(x)
\]

is the unique viscosity solution of the PDE

\[
\begin{align*}
\partial_t u &= \min \left( 0, \frac{\sigma^2}{2} \Delta u \right), \text{ on } (0, \infty) \times \mathbb{R} \\
u(0, \cdot) &= u_\mu(\cdot) - u_\nu(\cdot).
\end{align*}
\]

However, while the PDE gets simpler — in the sense that standard parabolic comparison results apply and existence follows from Perrons’ method — the question which conditions on \((\sigma, \mu, \nu)\) actually guarantee the existence of a Rost barrier which solves the Skorokhod embedding problem and which additional assumptions on the Root barrier guarantee its uniqueness is much less clear to us; i.e. we are not able give an analogue of Theorem 6 which applies to general \((\sigma, \mu, \nu)\). In fact, the existence of the reversed/Rost barrier has been studied only for certain cases by Rost [34] (as well as [31, 32, 33] and we point to forthcoming work \(^5\) of Cox et al. for more sufficient conditions).

5. Application I: Calculating the Root barrier via the Barles–Souganidis method

Theorem 6 allows to calculate the Root solution of the Skorokhod embedding problem, \( \{ R \} = \mathcal{R}_{\text{reg}}(\sigma, \mu, \nu) \), by calculating the viscosity solution of the obstacle problem (10) with initial data \( u_\mu \), barrier \( u_\nu \), and heat coefficient \( \sigma^2 \). One of the benefits of the viscosity approach is that one can resort to the Barles–Souganidis method [3, 4] when discussing numerical schemes. (The Feynman–Kac representation of the solution to (2) as the solution to the RFBSDE (13) serves a secondary purpose, namely, that one can make use of existing numerical methods for the RFBSDE to obtain an approximation of the Root barrier). In fact, due its relevance for optimal stopping problems there exists a vast literature on schemes for the obstacle PDE (18). While it falls outside the scope of this article to study numerics of the obstacle PDE (10) in full generality we still give two

\(^5\)Personal communication.
brief applications: firstly we show that the classic Barles–Souganidis convergence result can be easily adapted to give convergence when \( \mu, \nu \) have compact support, secondly, we discuss the case of general \( \mu, \nu \) but under the extra assumption \( \sigma = 1 \) (i.e. embedding into Brownian motion) in which the literature on viscosity solutions even provides good rates of convergence.

5.1. \( \mu \) and \( \nu \) of bounded support. The assumption of compact support implies (via Proposition 2) that \( (N^\mu)^\circ \subset [a, b] \) for some \( a, b \in \mathbb{R} \); hence (10) reduces to a PDE on a compact set with uniformly continuous \( \sigma, u_\nu, u_\nu \) data. Under the assumptions of Theorems 3 or 4 we have \( \{ u \} = V_\infty (\sigma, u_\nu, u_\nu) \) and the results in [3, 4, 1] (see also [17]) can be easily adapted to prove convergence of numerical schemes. We give a quick construction using an explicit finite differences scheme to approximate the solution of (2). On \( O_T := [0, T] \times [a, b] \) and setting \( h := (\Delta t, \Delta x) = \left( \frac{T}{N_T}, \frac{b-a}{N_T} \right) \) for \( N_T, N_x \in \mathbb{N} \) large enough we define the time-space mesh of points

\[
G_h := \{ t_n : t_n = n\Delta t, n = 0, 1, \ldots, N_T \} \times \{ x_j : x_j = a + j\Delta x, j = 0, 1, \ldots, N_x \}.
\]

Let \( B(O_T, \mathbb{R}) \) be the set of bounded functions from \( O_T \) to \( \mathbb{R} \) and \( BUC(O_T, \mathbb{R}) \subset B(O_T, \mathbb{R}) \) the subset of bounded uniformly continuous functions. Take \( \psi \in BUC(O_T, \mathbb{R}) \), we define its projection on \( G_h \) by \( \psi^h : O_T \to \mathbb{R} \) with \( \psi^h : [0, T + \Delta t] \times [a - \Delta x/2, b + \Delta x/2] \) as \( \psi^h(t, x) := \psi(t_n, x_j) \) when \( (t, x) \in (t_n, t_{n+1}) \times (x_j - \Delta x/2, x_j + \Delta x/2) \) for some \( n \in \{0, 1, \ldots, N_T\} \) and \( j \in \{0, 1, \ldots, N_x\} \); of course \( \psi^h \in B(O_T, \mathbb{R}) \). We denote the approximation to our solution \( u \in BUC(O_T, \mathbb{R}) \) of (2) by \( u^h \in B(O_T, \mathbb{R}) \). Below we use a standard finite difference scheme, i.e. we define the operator \( S^h : B(O_T, \mathbb{R}) \times [0, T] \times [a, b] \to \mathbb{R} \) as

\[
\begin{align*}
S^h [u^h] (t, x) &:= \begin{cases}
    u^h(x), & (t, x) \in [0, \Delta t) \times (a, b) \\
    u^h(t, x) + \frac{\Delta t \sigma^2}{2(2\Delta x)} (u^h(t, x + \Delta x) - 2u^h(t, x) + u^h(t, x - \Delta x)), & (t, x) \in [\Delta t, T] \times (a, b) \\
    u^h(t, x), & (t, x) \in [0, T] \times \{a, b\}
\end{cases}
\end{align*}
\]

where the well known CFL condition (computed for this case) \( CFL := \Delta t |\sigma|_{[a, b] \times [0, T]} < (\Delta x)^2 \) is assumed to hold. The values of \( u^h \) are computed by solving for \( u^h(t, x) \) via \( G(.) = 0 \) where \( G : (0, \infty)^2 \times O_T \times \mathbb{R} \times B(O_T, \mathbb{R}) \to \mathbb{R} \) is defined as

\[
G(h, (t + \Delta t, x), u^h(t + \Delta t, x), u^h) := \min \big\{ u^h(t + \Delta t, x) - u_\nu(x), u^h(t + \Delta t, x) - S^h[u^h](t, x) \big\}
\]

(for \( t + \Delta t, x \in O_T \) of course). Following [3, 4] we only have to guarantee that the operator \( S^h[\cdot] \) defined above and the PDE satisfies along some sequence \( h := (\Delta t, \Delta x) \) converging to zero the following properties

- **Monotonicity.** \( G(h, (t, x), r, f^h) \leq G(h, (t, x), r, g^h) \) whenever \( f \leq g \) with \( f, g \in \mathbb{B} \) (and for finite values of \( h, t, x, r \));
- **Stability.** For every \( h > 0 \), the scheme has a solution \( u^h \) on \( G_h \) that is uniformly bounded independently of \( h \) (under the CFL condition, see above);
- **Consistency.** For any \( \psi \in C^0_b(O_T; \mathbb{R}) \) and \( (t, x) \in O_T \), we have (under the CFL condition, see above):

\[
\lim_{(h, \xi, t_n + \Delta t, x_j) \to (0, 0, t, x)} \left( \psi(t_n + \Delta t, x_j + \xi) - u_\nu(x) \right) \leq \frac{\psi(t_n + \Delta t, x_j + \xi) - S^h[u^h + \xi](t_n, x_j)}{\Delta t} = \min \left\{ \psi(t, x) - u_\nu(x), \left( \partial_\xi \psi - \frac{\sigma^2}{2} \Delta \psi \right)(t, x) \right\}
\]

- **Strong uniqueness.** if the locally bounded USC (resp. LSC) function \( u \) (resp. \( v \)) is a viscosity subsolution (resp. supersolution) of (2) then \( u \leq v \) in \( O_T \);

**Proposition 6.** Let \( T \in (0, \infty) \). Assume \( \mu, \nu \) have compact support and that \( \sigma, \mu, \nu \) fulfill the assumptions of Theorem 6. Then \( u^h \in B([0, T] \times \mathbb{R}, \mathbb{R}) \) and

\[
|u^h - u|_{\infty, [0, T] \times \mathbb{R}} \to 0 \text{ as } h \to (0, 0)
\]

where \( u \) is the unique viscosity solution of (2) on \( [0, T], \{ u \} \in V_T(\sigma, \mu, \nu) \).

**Proof.** This results is more or less a direct consequence of [3, Theorem 2.1] and [1, Theorem 2.1]: the required strong uniqueness of the obstacle PDE (2) follows from the comparison theorem in the appendix while the existence from Theorem 5. The monotonicity, stability and consistency follow (stability and consistency conditions of the scheme hold under the CFL condition above) by a direct calculation (which we leave to the reader; see e.g. [4, 3, 1, 2]). The rest of the proof is a direct adaptation of the proofs [3, Theorem 2.1] and [1, Theorem 2.1] combined with the remarks on the first example in [1, Section 5]. One first shows that the operator \( S^h[\cdot] \) approximates indeed the PDE component of (2) then one adds in the barrier to recover the full equation (2). The core ingredients are the semi-relaxed limit technique of Proposition 4 combined with the strong uniqueness result (see Theorem 9) as sketched out in [1, p130].
Remark 13. An issue is the possibility of smoothing $u_\mu$ and $u_\nu$ prior to the numerical approximation. Usually $u_\mu, u_\nu$ have "kinks", for instance a common setup is with $\mu = \delta_0$ leading to $u_\mu(x) = -|x|$ with a kink at $x = 0$ or $\nu$ has atoms and these atoms are (as expected) a source of instability for finite difference schemes. Within the financial framework, the measure $\nu$ is given by market data, the observed prices of call options, via the

\[
\sigma(x) = \begin{cases} \sigma_0 & \text{for } x < 0 \\ \sigma_1 & \text{for } x > 0 \end{cases}
\]

where $\sigma_0 = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_0 + \frac{1}{2} \delta_{1}$ (hence $N^{\mu, \nu} = \mathbb{R} \setminus (-1, 1) \cup \{\pm \infty\}$). Applying above explicit finite difference scheme with CFL = 0.2 and $50K$ time steps on the interval $[0, 2]$ gives an approximation of (2) on the spatial domain $[-1, 1]$. The figure on the right shows that for $t_0 = 0.39348$ we have $u(t_0, 0) = u_\nu(0) = \frac{1}{2}$ which determines the spike at $x = 0$ for the Root barrier depicted on the right.
Breeden–Litzenberger formula. In practice the data available is only for finite amount of strikes and hence the inferred measure $\nu$ will have atoms. It seems intuitive that smoothing $u_\nu$ (or $\nu$) leads to a more stable method (see for example [19]). What is not clear is how to smooth $u_\nu$ (or $\nu$) without adding arbitrage.

5.2. Calculating general embeddings for Brownian motion ($\mu$ and $\nu$ are of unbounded support).

We now discuss the case of general $\mu, \nu$. For simplicity we restrict ourselves to embeddings into Brownian motion (i.e. $\sigma \equiv 1$) in which case more recent results of Jakobsen [22] directly apply and lead to a rate of order $1 2$ (if one is only interested in convergence the argument given above can be modified). Denote again $h = (\Delta t, \Delta x)$ and consider schemes of the type

$$u_h(t + \Delta t, x) = \max (S_{\Delta t} u_h(t, x), u_\nu(x))$$

where $S_{\Delta t}$ is the (formal) solution operator associated to the heat equation $\partial w - \frac{1}{2} \Delta w = 0$. In the case that we use a finite difference method to approximate $S$ the scheme can be written as

$$\min \left( \frac{u_h(t + \Delta t, x) - u_h(t, x)}{\Delta t} - \frac{u_h(t, x + \Delta x) - 2u_h(t, x) + u(t, x - \Delta x)}{(\Delta x)^2}, u_h(t + \Delta t, x) - u_\nu(x) \right) = 0.$$ 

A direct calculation shows that this is equivalent to

$$u_h(t + \Delta t, x) = \max \left( u_\nu(x), \sum_{z \in \Delta x^2} \left( 1 - \frac{\Delta t}{(\Delta x)^2} \right) u_h(t, x + z) \right).$$

This representation can then be used to prove the following

**Proposition 7.** Let $\mu \leq_{cx} \nu$, $\mu, \nu \in M^2_\mu$. There exists a unique $u_h$ solving (30). Further, if $\Delta t \leq (\Delta x)^2$, $u$ denotes the unique viscosity solution of (2) $(\{u\} = \mathcal{V}_T(1, u, u_\nu))$ and $u_{0,h}$ is an approximation of $u_0$ which is bounded independently of $h$ then

$$|u - u_h|_\infty \lesssim \sup_{[0, \Delta t] \times \mathbb{R}} |u - u_{0,h}| + (\Delta x)^{1/2}$$

**Proof.** This is a direct consequence of [22, Section 3]. The main idea is to show that one can replace the Barles–Souganidis assumptions by more special conditions ($C1 - C5$ in [22, Section 2]) which can be checked for the finite-difference scheme.

**Remark 14.** In any real life implementation one is forced to deal with a domain truncation procedure in order to approximate the solution, the convergence of the truncated PDEs has to be shown as well as existence/uniqueness of the solution to the problem in the space of discontinuous (due to truncation) viscosity solutions. Moreover, the truncation creates the appearance of an artificial boundary at the truncation points and in this boundary one can impose either Cauchy– or Neumann–type conditions for the approximation leading to non-trivial questions. Note that under the compact support assumption for $\mu, \nu$ treated in the first part even though one interprets the PDE in a bounded domain, $u_\mu$ and $u_\nu$ are known outside this domain hinting that one could use higher order finite differences stencils for the spatial derivatives without having to introduce an artificial boundary as in the usual case.

6. Application ii: model-independent bounds on variance options

We briefly recall an application of Root’s barrier to obtain model-independent bounds on variance options (which was Dupire’s [14] original motivation for looking at Root’s solution): much of financial mathematics is concerned with the pricing of claims contingent on the evolution of a risky asset. One approach is to

- Firstly, postulate a model for the risky asset, viz., denote with $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ a probability space satisfying the usual conditions carrying a real-valued process $P = (P_t)$ which describes the evolution of the price of a liquid asset and with constant interest rates $r$ the forward price process $S = (S_t)$ of this risky asset is given as

$$S_t = e^{-rt} P_t;$$

- Secondly, calculate an arbitrage–free price in this model as the expected value of the claim under a risk–neutral measure, viz., if $F_T = F(S_t : 0 \leq t \leq T)$ denotes the payoff of the claim, one considers a (so-called pricing) measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $S$ is a local martingale and an arbitrage–free price is then given as $E^\mathbb{Q} [F_T].$

Indeed, if $F_T$ equals $(S_T - K)^+$ or $(K - S_T)^+$ these claims form a liquid market and from observed prices one can in principle calibrate the chosen model. However, this approach falls short when $F_T$ is an exotic, non–liquid option and in practice a trader is faced with the following question: how to price a given

$$F_T = F(S_t : 0 \leq t \leq T)$$
(which might be highly path–dependent hence prices might be highly model–dependent) under the only assumption that \( S \) is a local martingale and that prices \( e(T, K) \) for \( F_T = (S_T - K)^+ \) are known for every \( T, K \).

Surprisingly there is a straightforward link with the Skorokhod embedding problem — see the excellent surveys of Obłój [28] and Hobson [21] — and we recall the argument in a nutshell:

- the market prices \( e(K,T) \) give (via the Breeden–Litzenberger formula) the distribution of the increment of the forward price, i.e
  \[
  \mu(dx) = \frac{d^2 e(T, x)}{dx^2} dx,
  \]
- any calibrated model \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) with forward price process \( S \) must have \( S_T - S_0 \sim \mu \),
- assuming Dambsis/Dubins–Schwarz applies we can write \( S - S_0 = W_{\tau(\cdot)} \) (\( W \) denoting a Brownian motion, \( \tau(\cdot) \) a time–change), hence
  \[
  W_{\tau(T)} \sim \mu
  \]
  and \( \tau(T) \) solves the Skorokhod embedding problem\(^6\).

From the above we conclude that all arbitrage–free prices of \( F_T \) are contained in
\[
\left[ \inf_{\tau} \mathbb{E} [ F(W_{\tau(t)} : 0 \leq t \leq T) ], \sup_{\tau} \mathbb{E} [ F(W_{\tau(t)} : 0 \leq t \leq T) ] \right]
\]
with sup and inf taken over all stopping times \( \tau \) which solve the Skorokhod embedding problem for \( \mu \), further, the bounds are attained by “extremal” solutions of the Skorokhod problem which in the case of variance options is given by the Root and Root solutions.

We now give a rigorous argument for the latter case of variance options: henceforth assume without loss of generality that \( S_0 = P_0 = 1 \) and that
\[
F_T = f([\ln P]_T) = f([\ln S]_T)
\]
where \( f \) is a convex, increasing function (e.g. \( f(x) = (-x - k)^+ \) for fixed \( k \geq 0 \)).

The interest in the Root model is that it gives the universal lower bound among all possible arbitrage–free forward price processes \( S \) with a prescribed marginals.

**Theorem 7.** Let \( S \) be a positive, real–valued continuous local martingale carried on some probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\) satisfying the usual conditions. Further assume, \( [\ln S]_\infty = \infty \) a.s. and
\[
S_T \sim \nu.
\]
Let \( R \in \mathcal{R}(id, \delta_1, \nu) \). Then for every \( f : [0, \infty) \to [0, \infty) \), \( f(0) = 0 \) which is convex and has a bounded right derivative,
\[
\mathbb{E}[f(\tau_R)] \leq \mathbb{E}^\mathbb{Q}[f([\ln S]_T)].
\]
Moreover, this bound is sharp in the sense that there exists a continuous local martingale \( S^R \) (which is independent of \( f \)) such that (31) becomes an equality.

**Proof.** It follows from Lemma 7 given below that there exists a Brownian motion \( B \) on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\) such that
\[
S_t = \exp \left( B_{[\ln S]_t} - \frac{[\ln S]_t^2}{2} \right).
\]
Since by assumption \( S_T \sim \nu \), and \((\mathcal{E}_{t \wedge [\ln S]_t})_t \) is u.i., it follows that \([\ln S]_T \) is a solution of the Skorokhod embedding problem \( S(id, \delta_1, \nu) \). Now using the minimality of the Root solution, Theorem 4, implies (31). To see that this bound can be attained consider a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) satisfying the usual conditions which carries a Brownian motion \( B \). Using the Root barrier \( R \) we define a process on this space
\[
S^R_t := \exp \left( B_{\tau_R \wedge \tau_R} - \frac{1}{2} \left( \tau_R \wedge \frac{T}{T-1} \right) \right)
\]
Then \( S^R \) is a continuous, local martingale with respect to the filtration \( \mathcal{F}^R_t := \mathcal{F}_{\tau_R}, S^R_T \sim \nu \) and
\[
[\ln S^R]_T = \tau_R.
\]
That it, \( S^R \) is an admissible pricing model which matches the marginals and \([\ln S^R]_T \) attains the lower bound. \( \square \)

**Lemma 6.** A continuous, positive process \( S \) on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\) is a continuous local martingale \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\)

\[
S = S_0 \exp \left( M - \frac{1}{2} [M] \right) \ a.s.
\]
for some continuous, local martingale \( M \). In that case \( M = \int_0^T S^{-1} dS \) a.s.

\(^6\)To make this step rigorous needs a bit more care since there might not exists such a time change etc.
**Proof.** Follows from a direct application of the Itô formula to \( \ln S_t \).

**Lemma 7.** Let \( S \) be a positive, continuous local martingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\). Further assume \([\ln S]_\infty = \infty\) a.s. Then there exists a Brownian motion \( B \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\) such that

\[
S = S_0 \exp \left( B_{[\ln S]} - \frac{[\ln S]}{2} \right) \mathbb{Q} \text{ a.s.}
\]

**Proof.** Let \( M \) be the continuous local martingale \( M \) of Lemma 6. Since \([\ln S] = \ln S_0 + M - \frac{1}{2} [M] \) we have \([M]_\infty = [\ln S]_\infty = \infty\) a.s. By the Dambis/Dubins–Schwarz Theorem (cf. [29, Chapter 5, Theorem 1.7]) there exists a Brownian motion \( B \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\) s.t. \( M_t = B_{[M]} \forall t \mathbb{Q}\text{ a.s.} \)

**Remark 15.** The assumption \([\ln S]_\infty = \infty\) is needed for the Dambis/Dubins–Schwarz Theorem to guarantee the existence of a Brownian motion \( B \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\). It can be dispensed with for the price of using an enlargement of \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\) c.f. [29, Chapter 5, Theorem 1.7].

**Lemma 8.** Equip \([0,1] \times [1,1] \) with Euclidean metric and denote with \( r \) the metric on \([0,\infty] \times [0,\infty] \) induced by the homeomorphism

\[
[0,\infty] \times [-\infty, \infty] \rightarrow [0,1] \times [1,1] \quad (t,x) \rightarrow \left( \frac{t}{1+t}, \frac{x}{1+|x|} \right).
\]

Then,

1. the map \( \rho (R,Q) := \sup_{x \in Q} r(x,R) \vee \sup_{x \in R} r(x,Q) \)

is a metric on the space \( C \) of closed subsets of \([0,\infty] \times [-\infty, \infty] \).

2. \( (C, \rho) \) is a separable, compact metric space and \( R \) is a closed (hence compact) subset.

**Proof.** Root [30] (see also Loynes [26]).

With this metric, the map that associates a barrier to its hitting stopping time is uniformly continuous from \((R, r)\) into the set \((T, \text{c.p.})\) of first hitting to barriers equipped with the convergence in probability (c.p.). The next lemma states this more precisely.

**Lemma 9.** Let Assumption 1 hold. Let \( B \in \mathcal{R} \) with corresponding first hitting time \( \tau_B \). If \( \mathbb{P}[\tau_B < \infty] = 1 \), then \( \forall \epsilon > 0 \exists \delta > 0 \) s.t. \( \mathbb{P}[\tau_B > \delta] < \epsilon \).

Moreover, take a sequence of barriers \((R_n)\) and the associated hitting times \((\tau_{R_n})\), if the sequence of barriers converges \( (w.r.t. \) the metric \( r) \), \( R_n \rightarrow_n R \) and (defining \( \tau_R \) is the hitting time associated to \( R \) and) if \( \sup_n \mathbb{E}[\tau_{R_n}] = K < \infty \), then it holds that \( \mathbb{E}[\tau_{R_n}] \leq K \) and \( \mathbb{P}[|\tau_{R_n} - \tau_R| > \epsilon] \rightarrow_n 0 \).

**Proof.** The proof is a direct reformulation of the proofs of Lemma 2.4 of [30] and Lemma 1 of [26] which were stated with the Brownian motion being the underlying time-space process. Just remark that under Assumption 1 the solution of (6) has continuous paths, the strong Markov property, no interval of constancy and covers the required state space.

We can now start proving the main theorems.

---

1 An excellent article is Rost’s [34] but he just treats Markov processes taking values in a compact space.
7.1. Existence and uniqueness of a Root barrier if \([X]_\infty = \infty\). We prepare the proof of Theorem 3 with a lemma.

Lemma 10. Let assumption 1 hold and that \(\exists N \in \mathbb{N}\) s.t. \(\nu|_{\{m-N,m+N\}} = \sum_{i=1}^{n} p_i \delta_{x_i}\) for some \(n \in \mathbb{N}\) and

\[
\nu ((-(\infty, m-N)) = \mu (-(\infty, m-N)) \quad \text{and} \quad \nu ((m+N, \infty)) = \mu ((m+N, \infty))
\]

Then \(|\mathcal{R}_{reg}(\sigma, \mu, \nu)| = 1\) and \(\mathbb{E} [X]_\tau_c = \int_{\mathbb{R}} x \, d\nu (dx)\).

Proof. Define \(I_N := [m-N, m+N] \cap \mathbb{R}\) and define \(I_N^c := \mathbb{R}\setminus \int_{\mathbb{R}} x \, d\nu (dx)\). By the definition of \(\nu\) in \(I_N\) that (at least)

\[
u_\nu = u_\mu \quad \text{on} \quad I_N^c.
\]

In fact (32) first implies the equality in \(\mathbb{R}\setminus I_N\) which is then extended to \(I_N^c\) using the continuity of \(u_\mu, u_\nu\). We can then conclude that \(I_N^c \subseteq N^{\mu, \nu}\). By assumption \(\sum_{i=1}^{n} p_i = 1 - \nu (-(\infty, m-N)) - \nu ((m+N, \infty))\). Since we are looking for a Root barrier, it is clear that \(X\) should only stop at the support points \(\{x_i: 1 \leq i \leq n\}\) of \(\nu\) in \(I_N\) (else the stopped process would put mass at a point that is not in the support of \(\nu|_{I_N}\)). Take \(N^{\mu, \nu}\) and define the two indices sets \(I = \{1, \cdots, n\} \setminus \mathcal{J}\) and \(\mathcal{J} = \{j \in \{1, \cdots, n\}: x_j \in N^{\mu, \nu}\}\). For some (unknown) \(b = (b_1, \cdots, b_n) \in [0, \infty)^n\) with \(b_j = 0\) for any \(j \in \mathcal{J}\), we are then looking for a Root barrier \(R_b\) of the form

\[
R_b = \left( \bigcup_{i \in \mathcal{J}} [b_i, +\infty) \times \{x_i\} \right) \bigcup N^{\mu, \nu}.
\]

It is clear that if \(\mathcal{I} = \emptyset\) (implying that \(b = 0\)) then the embedding is trivial since \(u_\nu = u_\mu\) if \(\nu = \mu\).

Every such barrier \(R_b\) gives rise to a first hitting time \(\tau_b\) of the time-space process \((t, X_t)_{t \geq 0}\)

\[
\tau_b := \inf \{t \geq 0 : (t, X_t) \in R_b\}.
\]

By assumption \([X]_\infty = \infty\) which implies \(\tau_b < \infty\) a.s. since \(\tau_b\) is dominated by the first exit time of \(X\) of the interval \([x_1, x_n]\) (which if finite a.s. since \([X]_\infty = \infty\) implies \(\lim \sup_{t \to \infty} X_t = \infty\) and \(\lim \inf_{t \to \infty} X_t = -\infty\) see for example [29, Proposition V.1.8]).

The subset of barriers \(\Gamma_\nu\): Consider \(c \in [0, \infty)^n\) with a corresponding Root barrier \(R_c\) resp. hitting time \(\tau_c\).

Define

\[
\Gamma_\nu \equiv \{c \in [0, \infty)^n: c_j = 0 \quad \forall j \in \mathcal{J} \quad \text{and} \quad \mathbb{P} [X_{\tau_c} = x_i] = p_i \quad \forall i \in \mathcal{I}\}
\]

(for \(c \in \Gamma_\nu\), \(X_{\tau_c}\) embeds "less mass" at points \(x_i, i \in \mathcal{I}\), than required by \(\nu\)). Note that if \(c \in \Gamma_\nu\) then one has \(\sum_{j \in \mathcal{J}} \mathbb{P} [X_{\tau_c} = x_i] \geq \sum_{j \in \mathcal{J}} p_j\) since one never embeds more mass (under \(\nu\)) than required at any point in the interior \((N^{\mu, \nu})^c\) (i.e. when \(u_\mu \neq u_\nu\), consequently the excess mass must be embedded at the boundary and singular points of \(N^{\mu, \nu}\), viz. the points \(x_j\) with \(j \in \mathcal{J}\)).

\(\Gamma_\nu\) has a minimal element \(\gamma\) (in \(\Gamma_\nu\)): We argue as in [21]. We claim that \(\Gamma_\nu\) with the binary relation "\(\leq\)" (component wise) is a lower semi-lattice, i.e. (any finite subset of \(\Gamma_\nu\) has a minimal element in \(\Gamma_\nu\))

\[
\text{if} \quad b, c \in \Gamma_\nu \quad \text{then} \quad \gamma = b \land c := (b \land c)_i = \min(b_i, c_i) \quad \forall i \in \mathcal{I}.
\]

To see this, fix \(i \in \{1, \cdots, n\}\) (for \(i \in \mathcal{J}\) the argument is trivial as \(b_i = c_i\) and assume wlog that \(b_i \leq c_i\). Then \(\tau_\gamma = \tau_\gamma\) on \(X_{\tau_\gamma} = x_i\) and \(\tau_\gamma \leq \tau_\gamma\) otherwise. Since the barrier points \(\gamma\) are the smallest of the corresponding points between \(b\) and \(c\), the hitting time of \(\gamma\) must be smaller or equal than the hitting times of \(b\) and \(c\). Thus, \(\{X_{\tau_\gamma} = x_i\} \subset \{X_{\tau_\gamma} = x_i\}\) and \(\mathbb{P} [X_{\tau_\gamma} = x_i] \leq p_i\). The statement follows for general \(i \in \{1, \cdots, n\}\).

It is clear that the space of barriers of the form of (33) has a minimal element \((b = 0)\) and being \(\Gamma_\nu\) a lower semi-lattice it is also clear that an infimum, say \(d\), to \(\Gamma_\nu\) exists (wrt to the partial order relation \(\leq\)); it may happen though that \(d \notin \Gamma_\nu\). Let \(\tau_d\) be the first hitting time of \(R_d\), we now prove that \(\tau_d \in \Gamma_\nu\): if \(d\) is the infimum to \(\Gamma_\nu\) then there exists a monotonic decreasing sequence \((d_k)_{k \in \mathbb{N}}\) of elements in \(\Gamma_\nu\) converging to \(d\). The stopping times \(\tau_{d_k}\) induced by the elements \(d_k\) are uniformly (in \(k\)) dominated by the almost surely finite first exit time of the barrier \(N^{\mu, \nu}\). Using now Lemma 9 we can conclude that \(\tau_{d_k} \to \tau_d\) in probability (and hence has an a.s. converging subsequence). By the definition of \(\Gamma_\nu\) (see (34)) it follows that

\[
\forall i \in \mathcal{I} \quad \lim_{k \to \infty} \mathbb{P} [X_{\tau_{d_k}} = x_i] \leq p_i \Rightarrow \mathbb{P} [X_{\tau_d} = x_i] \leq p_i \quad \Rightarrow \quad d \in \Gamma_\nu
\]

where the implication follows by the convergence of \(\tau_{d_k} \to \tau_d\) (a.s. and in Probability) and the continuity of the paths of \(X\).

The minimum of \(\Gamma_\nu\) embeds \(\nu\): Set \(\gamma\) as the minimal element \(\gamma = (\gamma_1, \cdots, \gamma_n) \in \Gamma_\nu\) (wrt to the partial order \(\leq\)) of \(\Gamma_\nu\) and associate the induced barrier \(R_\gamma\) and its hitting time \(\tau_\gamma\). We now show that \(\gamma\) embeds \(\nu\) via \(X\). To see this assume \(X_{\tau_\gamma} \sim \nu\). In this case

\[
\exists i \in \mathcal{I} \quad \text{s.t.} \quad \mathbb{P} [X_{\tau_\gamma} = x_i] < p_i
\]

but this contradicts the minimality of \(\gamma\) since we can find \(\tau \in [0, \infty)^n\) by setting \(\tau_k = \gamma_k\) for \(k \neq i\) and \(\tau_i = \gamma_i - \epsilon\) where \(\epsilon > 0\) is chosen small enough s.t. \(\mathbb{P} [X_{\tau} = x_i] \leq p_i\). Then \(\tau \in \Gamma_\nu\) since \(R_{\tau} \supset R_\gamma\) with the only difference between the two barriers being the point \(\gamma_i - \epsilon\) instead of \(\gamma_i\). That is, we have increased the probability paths of
Proof. The existence of such a sequence is easy as it follows a construction of the potential functions $X_\gamma$. We can now prove Theorem 3. In the case $\nu(x) = 0$ for $x > 0$ one can approximate $\nu(x)$ by a sequence of probability measures $(\nu_N)_{N \in \mathbb{N}}$ with finite support in a certain region and that converges to $\nu$ in weakly. We first state a lemma on the existence of the approximating sequence where each measure satisfies conditions on the line of that in Lemma 10.

**Lemma 11.** Let $m \in \mathbb{R}$, $\mu, \nu \in \mathcal{M}_m^2$ verifying $\mu \leq_{cs} \nu$. Then there exists a sequence of measures $(\nu_N)_{N \in \mathbb{N}} \subset \mathcal{M}_m^2$ s.t.

1. $\nu_N$ converges weakly to $\nu$ as $N \to \infty$,
2. $\nu_N(m - N, m + N) = \sum_{i=1}^{N} \mu(\delta_{x_i})$ for some $(x_i)^N_{i=1} \subset (m - N, m + N)$,
3. $\nu_N \leq_{cs} \nu_N \leq_{cs} \nu$.

**Proof.**

The existence of such a sequence is easy as it follows a construction of the potential functions $u_{\nu_N}$ similar to Dubin's construction (see p332–333 in [27]) then via point (5) of Proposition 2 it follows that $\nu_N \Rightarrow \nu$. Fix $N$ and have in mind the potential picture of $\mu, \nu$. Now apply a Chacon-Wash like potential construction but starting from $u_\mu$ instead of $u_\delta$, using, $4^N$ lines that includes forcefully the points $(m - N, u_\mu (m - N))$ and $(m + N, u_\mu (m + N))$. The obtained function is $u_{\nu_N}$.

By construction $u_{\nu_N}$ is a true potential function (simply follow the arguments in [9]), it coincides with $u_\mu$ in $I_N$ and $u_{\nu_N} \leq u_\mu \leq u_{\nu_N}$ in $J_N$. Moreover, the measure $\nu_N$ induced by $u_{\nu_N}$ (see Proposition 1) is of finite support in $I_N$ hence $\nu_N \in \mathcal{M}_m^2$ as well as $\nu_N \leq_{cs} \nu_N \leq_{cs} \nu$. It is also clear that as $\nu_{\nu_N} \to_N u_\nu$ and hence $\nu_N \Rightarrow \nu$ (see Proposition 2).

We can now prove Theorem 3. In the case $\mathcal{R}(1, \delta_0, \nu)$ (see [30]) one simply approximates $\mu$ by a sequence of finitely supported measures $(\mu_N)_n$ but for our more general case this is not possible (unless one approximates as well the initial measure which makes the proofs much more involved).

**Proof of Theorem 3.** For $N \in \mathbb{N}$ define $I_N := [m - N, m + N]$ and $I_N := \mathbb{R} \setminus (I_N)^c$. Take a sequence $(\nu_N)_{N \in \mathbb{N}}$ of probability measures such that $\nu_N \Rightarrow \nu$ and where the measures $(\nu_N)_{N}$ are as in Lemma 11.

It is clear that each $\nu_N$ fulfills the assumptions of Lemma 10 and hence $\mathcal{R}_{reg}(\sigma, \mu, \nu) = \{N\}$ and we denote with $\tau_N$ the corresponding hitting time (minimal and uniformly integrable s.th. $X_{\tau_N} \sim \nu_N$ and $\mathbb{E}[X_{\tau_N}] = \mathbb{E}[X_{\tau_N}] = \mathbb{E}[\hat\nu_N(dx)] \leq \mathbb{E}[\nu(dx)] < \infty$). The compactness of $(\mathcal{R}, \rho)$ guarantees the convergence of $(\mathcal{R}_N)_{N} \subset \mathcal{R}$ along some subsequence to some $\mathcal{R} \in \mathcal{R}$. The problem setup and the embedding properties yield

$$\sup_{N \in \mathbb{N}} \mathbb{E}[X_{\tau_N}] \leq K < \infty \quad \text{and} \quad [X]_{\infty} = \infty.$$ 

The continuity of the quadratic variation process implies $\sup_{N \in \mathbb{N}} \mathbb{E}[\tau_N] \leq \tilde{K} < \infty$ for some $\tilde{K} > 0$ (otherwise there would exist a subsequence $(\tau_N)_N$ with $\mathbb{E}[\tau_N] \to \infty$ and therefore $\mathbb{E}[X_{\tau_N}] = \mathbb{E}[\hat\nu_N(dx)] > K$). By the second part of Lemma 9 we conclude that $\mathbb{E}[\tau]\leq \tilde{K} < \infty$ (as well as $\mathbb{E}[X_{\tau}] \leq K < \infty$) and that $\tau_N \to_N \tau$ in probability.

It remains to show that $X_{\tau_N} \sim \nu$. Now $\forall N \in \mathbb{N}$ and $\forall f \in C_b$

$$\int_{\mathbb{R}} f(x) \nu_N(dx) = \mathbb{E}[f(X_{\tau_N})]$$

and by the above there exists a subsequence $(\tau_N)_N$ s.t. $\tau_N \to_N \tau$ a.s.. Dominated convergence together with continuity of the sample paths of $X$ (and $f$) along with $\nu_N \Rightarrow \nu$ show that in the limit $\forall f \in C_b$

$$\int_{\mathbb{R}} f(x) \nu(dx) = \mathbb{E}[f(X_{\tau})]$$

or equivalently that $X_{\tau_N} \sim \nu$ and hence $\mathcal{R} \in \mathcal{R}_{reg}(\sigma, \mu, \nu)$. The uniqueness of $\mathcal{R}$ follows from Theorem 2.

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*For a more graphical construction with detailed explanation, of which Dubin’s solution is a particular case, we point the reader to the Chacon-Wash embedding [9]; see alternatively p337–339 in [27].
7.2. Existence and uniqueness of a Root barrier if $X$ is a GBM. If $\sigma(t,x) = x$, the process $X$ is a GBM which implies $[X]_\infty < \infty$ and $[\ln X]_\infty = \infty$ (this follows for example by Dambis/Dubins–Schwarz), hence the results of the previous section do not apply immediately to this important case. Nevertheless, some simple modifications allow for a similar argumentation. We prepare the proof by noting that Lemma 9 also applies to Geometric Brownian motion.

Lemma 12. Let $\sigma(x) = x$, then the conclusions of Lemma 9 remain valid.

Proof. The proof is a direct reformulation of the proofs of Lemma 2.4 of [30] and Lemma 1 of [26] which were stated with the Brownian motion being the underlying time-space process. □

We can now prove Theorem 4.

Proof of Theorem 4. The proof follows the same method as the proof of Theorem 3. Assume $m = 1$. It is clear that Lemma 10 holds for this setting (with the proper adaptation of the proof). We point from Assumption 2 that the support of $\mu$ is bounded from below by a number $\epsilon$. This is crucial as it implies that the hitting time to any Root barrier (in this setting) will be dominated by the first exit time of the GBM to the interval $[\epsilon, \infty)$.

Define a sequence of approximating probability measures $(\nu_N)_{N \in \mathbb{N}}$ such that $\nu_N \Rightarrow \nu$ and define the sequence in the same fashion as in the proof of Theorem 3, s.th. $\nu_N$ is finitely supported in $[\epsilon, N]$ and outside it it coincides with $\mu$. Since each $\nu_N$ is finitely supported in $[\epsilon, N]$ (outside it we have $u_{\mu} = u_{\nu_N}$) Lemma 10 yields the existence of a barrier $R_N$ and its (uniformly integrable) corresponding first hitting time $\tau_{N}$ s.th. $X_{\tau_{N}} \sim \nu_{R}$ and $\mathbb{E}[[X]_{\tau_{N}}] = \int x^2\nu_N(dx) \leq K < \infty$ with $K > 0$.

From compactness of the barrier space $(R, r)$, we can extract a converging subsequence of $(R_N)_{N \in \mathbb{N}}$ with limit, say $R$, and let $T_R$ be its corresponding first hitting time (we abuse notation and still denote by $(R_N)_{N}$ the converging subsequence). Now remark that all the stopping times $T_N$ are dominated by the stopping time $\tau$, the first time of $(X_t)_{t \geq 0}$ to the interval $[\epsilon, \infty)$ (see the observations above). Since $\tau$ is a.s. finite then so is the sequence $T_N$ uniformly dominated and hence Lemma 12 can be applied. The rest of the passage to the limit argument follows like the proof of Theorem 3. □

Remark 17. If $\text{supp}(\nu) \subset [0, \infty)$ (with $\nu(\{0\}) = 0$) then above argument does not ensure the uniform boundedness of $(T_N)$ via the uniform boundedness of the $L^2$-norms of $[X]_{\tau_N}$ and this is the reason for Assumption 2 (i.e. $\text{supp}(\nu) \subset [\epsilon, \infty)$ with $\epsilon > 0$).

7.2.1. Optimality of Root’s solution for Geometric Brownian motion. For the general case of embedding a probability distribution via an Itô process of the form (6) where $\sigma$ satisfies Assumption 1 with the additional assumption of time-homogeneity along with Assumption 2 for $\mu$, $\nu$ the optimality result has already been stated and proved in Root [34] and Cox–Wang [11]. The particular case of GBM follows also from [11, Theorem 5.3, Remark 5.5]. We recall their result: in its due generality, Root’s construction solves the following general problem, as under Assumption 2 the state space of $X$ will be in $[\epsilon, \infty)$, hence $\sigma > 0$. In this case it is possible to show that the uniform integrability of the stopping time is equivalent to the minimality (in the sense of Definition 2). See e.g. Section 8 in [27] or Section 3.4 in [21].

Minimize: $\mathbb{E}[F(\tau)]$

subject to: $X_{\tau} \sim \nu$

$\tau$ a UI stopping time.

Here we work under the assumptions of Theorem 4 and $F$ is a given convex, increasing function with bounded right derivative $f$ and satisfying $F(0) = 0$. Under the assumptions of Theorem 4 there exists a Root Barrier $R$ and its corresponding stopping time $\tau_R$ which perform the embedding. Define the function

$$M(t,x) := \mathbb{E}^{(t,x)}[f(\tau_R)], \quad (t,x) \in [0, +\infty) \times (0, +\infty)$$

as well as

$$Z(x) := 2 \int_{1}^{x} \int_{1}^{y} \frac{M(0,z)}{\sigma^2(z)} dzdy \quad x \in (0, +\infty).$$

Theorem 8 (Cox–Wang [11], Optimality of the Root embedding). Let the assumptions of Theorem 4 hold. Let $R \in \mathbb{R}(\sigma, \mu, \nu)$ and $F$ be any given convex, increasing function satisfying $F(0) = 0$ and with (bounded) right derivative $f(x) := F'(x)$. Assume further that (36) is locally bounded on $[0, \infty) \times (0, +\infty)$ and that for any $T > 0$ the function (37) satisfies

$$\mathbb{E} \left[ \int_{0}^{T} |Z'(X_s)\sigma(X_s)|^2 ds \right] < \infty \text{ and } \mathbb{E}[Z(X_0)] < \infty.$$
Then for any stopping time $\tau$ for which $X_\tau \sim \nu$ it holds that

$$\mathbb{E}[F(\tau_R)] \leq \mathbb{E}[F(\tau)].$$

Proof. See Theorem 5.3 and Remark 5.5 in [11]. For the particular setting of GBM with initial condition $X_0 \sim \delta_1$ both (36) and (38) are easily satisfied, see [37, Theorem 4.3.4].

8. Appendix: Comparison theorem

Comparison theorems for obstacle problems can be found in the literature, see [15] or [23, 22]. However, due to the unboundedness of the coefficients they do not cover immediately our setup. For the convenience of the reader we provide a complete proof by adapting some results found in [23, 22, 13]. Further, it shows the Hölder regularity in space of viscosity solutions (which is the first step to get convergence rates for numerical schemes).

**Theorem 9.** Let $h \in C(\mathbb{R}, \mathbb{R})$ be of linear growth, i.e. $\exists c > 0$ such that

$$|h(x)| \leq c(1 + |x|) \quad \text{for all } x \in \mathbb{R}$$

and $\sigma \in C([0, T] \times \mathbb{R}, \mathbb{R})$ Lipschitz in space, uniformly in time ($\sup_t |\sigma(t, \cdot)|_{Lip} < \infty$). Define

$$F_{obs}(t, x, r, a, p, M) = \min\left( u(t, x) - h(x), a - \frac{\sigma^2(t, x)}{2}m \right).$$

Let $u \in USC([0, T] \times \mathbb{R}, \mathbb{R})$ be a viscosity subsolution and $v \in LSC([0, T] \times \mathbb{R}, \mathbb{R})$ a viscosity supersolution of the PDE

$$F_{obs}(t, x, \partial_t u, Du, D^2 u) \leq 0 \leq F_{obs}(t, x, \partial_t v, Dv, D^2 v) \text{ on } (0, T) \times \mathbb{R}$$

Further assume that $\forall (t, x) \in [0, T] \times \mathbb{R}$, $u(t, x), -v(t, x) \leq C(1 + |x|)$ for some constant $C > 0$ and that $u(0, \cdot)$ and $v(0, \cdot)$ are $\delta$-Hölder continuous. Then there exists a constant $c \geq 0$ s.t. $\forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$

$$u(t, x) - v(t, y) \leq \sup_{z \in \mathbb{R}} (u_0(z) - v_0(z)) + c \inf_{\alpha > 0} \left\{ \alpha^{-\frac{1}{2+\delta}} + |x - y|^2 \right\}.$$  

Direct consequences of this estimate are

1. $u_0 \leq v_0$ implies $u \leq v$ on $[0, T] \times \mathbb{R}$,
2. if $u$ is also a supersolution (viz. $u$ is a viscosity solution) then $u$ is $\delta$-Hölder continuous in space uniformly in time on $[0, T)$, i.e.

$$\sup_{t \in [0, T]} |u(t, \cdot)|_{C^{\delta}(\mathbb{R})} < \infty.$$

Proof. Wlog we can replace the parabolic part in $F$ with $\frac{\partial w}{\partial t} - \sigma^2 \Delta w - w$ (by replacing $u$ resp. $v$ with $e^{-t}u$ resp. $e^{-t}v$). Further we can assume that $\forall \tau > 0$, $u$ is a subsolution of

$$F_{obs}(t, x, \partial_t u, Du, D^2 u) \leq -\frac{\tau}{(T - t)^2} \quad \lim_{t \uparrow T} u(t, x) = -\infty \text{ uniformly on } \mathcal{O}$$

(by replacing $u$ with $u - \frac{\tau}{(T - t)}$). Define for $\alpha > 0, \epsilon > 0$

$$\psi(t, x, y) = u - v - \phi(t, x, y) \quad \text{with} \quad \phi(t, x, y) = e^{\lambda t} \alpha |x - y|^2 + \epsilon \left(|x|^2 + |y|^2\right)$$

and

$$m_{0, \alpha, \epsilon} = \sup_{x, y \in \mathbb{R}} \psi(0, x, y) + \sup_{t \in [0, T] \times \mathbb{R} \times \mathbb{R}} \psi(t, x, y) - m_{\alpha, \epsilon}^0.$$  

The growth assumptions on $u$ and $v$ together with (39) guarantee for every $\alpha > 0, \epsilon > 0$ the existence of a triple $(\hat{t}, \hat{x}, \hat{y}) \in [0, T) \times \mathbb{R} \times \mathbb{R}$ s.t.

$$m_{\alpha, \epsilon} + m_{0, \alpha, \epsilon} = \psi(\hat{t}, \hat{x}, \hat{y}).$$

The proof strategy is classic: the above implies that $\forall \alpha > 0, \epsilon > 0$ and $\forall (t, x, y)$

$$u(t, x) - v(t, y) \leq m_{\alpha, \epsilon}^0 + e^{\lambda t} \alpha |x - y|^2 + \epsilon \left(|x|^2 + |y|^2\right).$$

Using the Hölder continuity of $u_0$ and $v_0$ we immediately get an upper bound for $m_{\alpha, \epsilon}^0$

$$m_{\alpha, \epsilon}^0 \leq u_0(\hat{x}) - v_0(\hat{x}) + |v_0|^\delta |\hat{x} - \hat{y}|^\delta - \alpha |\hat{x} - \hat{y}|^2$$

and below we use the parabolic theorem of sums to show that

$$m_{\alpha, \epsilon} \leq C\alpha^{-\frac{1}{2+\delta}} + k\epsilon + \omega_{\alpha}(\epsilon).$$
where $\omega_\alpha(\cdot)$ is a modulus of continuity for every $\alpha > 0$. Plugging these two estimates into (40) gives

$$u(t, x) - v(t, y) \leq |u_0 - v_0| + (c + C) \alpha^{-\frac{1}{2\sigma}} + e^{\lambda t} \alpha |x - y|^2 + \epsilon \left( |x|^2 + |y|^2 \right).$$

Now letting $\epsilon \to 0$ and subsequently optimizing over $\alpha$ yields the key estimate

$$u(t, x) - v(t, y) \leq |u_0 - v_0| + \inf_{\alpha > 0} \left( (c + C) \alpha^{-\frac{1}{2\sigma}} + e^{\lambda t} \alpha |x - y|^2 \right).$$

Applying it with (42) such that

$$\omega_{\alpha, \epsilon}$$

Now assume the first term in the (43) hence we get the estimate

$$\alpha$$

It remains to show (41). Below we assume $m_{\alpha, \epsilon} \geq 0$ and derive the upper bound (41) (which then also holds if $m_{\alpha, \epsilon} < 0$). Note that $m_{\alpha, \epsilon} \geq 0$ implies $\ell > 0$. The parabolic Theorem of sums [12, Theorem 8.3] shows existence of

$$(a, D_x\psi (\hat{t}, \hat{x}, \hat{y}), X) \in \mathfrak{P}^{2,+} u (\hat{t}, \hat{x})$$

and subtracting the second inequality from the first leads to

$$\min \left( a - \frac{\sigma^2 (\hat{t}, \hat{x})}{2} X - u (\hat{t}, \hat{x}), u (\hat{t}, \hat{x}) - h (\hat{x}) \right) \leq 0$$

and subtracting the second inequality from the first to arrive at

$$\min \left( a - \frac{\sigma^2 (t, x)}{2} X - u (\hat{t}, \hat{x}), u (\hat{t}, \hat{x}) - h (\hat{x}) \right) \geq 0$$

First assume the second term in the min is less than or equal to 0. This gives

$$u (\hat{t}, \hat{x}) - v (\hat{t}, \hat{y}) \leq h (\hat{x}) - h (\hat{y}) \leq |h| \hat{x} - \hat{y})$$

hence we get the estimate

$$m_{\alpha, \epsilon} \leq |h| |\hat{x} - \hat{y}|.$$
Hence we have shown that \( \forall \alpha > 0 \):

\[
\limsup_{\epsilon} m_{\alpha, \epsilon} = C \alpha^{-\frac{1}{\alpha}} ,
\]

and

\[
u(t, x) - v(t, y) \leq m_{\alpha, \epsilon} + m_{\alpha, \epsilon}^0 + e^{\alpha M_t} \frac{1}{2} |\hat{x} - \hat{y}|^2 + \epsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right) .
\]

\[\square\]

Lemma 13. Let \( f \in USC \left( [0, T] \times \mathbb{R} \times \mathbb{R} \right) \) and bounded from above. Set

\[
m := \sup_{[0, T] \times \mathbb{R} \times \mathbb{R}} f(t, x, y),
\]

\[
m_{\epsilon} := \sup_{(t, x, y)} f(t, x, y) - \epsilon \left( |x|^2 + |y|^2 \right) \text{ for } \epsilon > 0 .
\]

Denote with \( (\hat{t}, \hat{x}, \hat{y}) \) points where the sup is attained. Then

1. \( \lim_{\epsilon \to 0} m_{\epsilon} = \sup_{[0, T] \times \mathbb{R} \times \mathbb{R}} f(t, x, y) \),
2. \( \epsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right) \rightarrow 0 \).

Proof. By definition of a supremum there exists for every \( \eta > 0 \) a triple \((t_\eta, x_\eta, y_\eta)\) \( \in [0, T] \times \mathbb{R} \times \mathbb{R} \) such that \( f(t_\eta, x_\eta, y_\eta) > m - \eta \). Fix \( \eta > 0 \) and take \( \epsilon' \) small enough s.t. \( \epsilon' \left(|x_\eta|^2 + |y_\eta|^2\right) \leq \eta \). Then \( \forall \epsilon \in [0, \epsilon'] \) we have

\[
m \geq m_{\epsilon} \geq f(t_\eta, x_\eta, y_\eta) - \epsilon' \left(|x_\eta|^2 + |y_\eta|^2\right) \geq f(t_\eta, x_\eta, y_\eta) - \eta \geq m - 2\eta .
\]

Since \( \eta \) can be arbitrary small and \( \epsilon \rightarrow m_{\epsilon} \) is non-increasing the first claim follows. From the above estimate and the boundedness of \( f \) from above also show that

\[
k_{\epsilon} = \epsilon \left(|\hat{x}|^2 + |\hat{y}|^2\right) \text{ is bounded. Hence there exists a subsequence of } (k_{\epsilon})_{\epsilon > 0} \text{ which we denote with slight abuse of notation again as } (k_{\epsilon})_{\epsilon > 0} \text{ which converges to some limit denoted } k(\geq 0) \text{. Now}
\]

\[
f(\hat{t}, \hat{x}, \hat{y}) - k_{\epsilon} \leq m - k_{\epsilon}
\]

and from the first part we can send \( \epsilon \) to 0 along the subsequence and see that \( m - k \leq m \), hence \( k = 0 \). Since we have shown that every subsequence \( (k_{\epsilon}) \) converges to 0 the second statement follows. \( \square \)

References


