On the convergence of finite state mean-field games through Γ-convergence

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Abstract

In this paper we study the long time convergence (trend to equilibrium problem) for finite state mean-field games using Γ-convergence. Our techniques are based upon the observation that an important class of mean-field games can be seen as the Euler-Lagrange equation of a suitable functional. Therefore, by a scaling argument, one can convert the long time convergence problem into a Γ-convergence problem. Our results generalize previous results on long-time convergence for finite state problems.

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1 Introduction

Mean-field games were developed in the engineering community by Peter Caines and his co-workers [HMC06, HCM07] and independently, at about the same time, by Pierre Louis Lions and Jean Michel Lasry [LL06a, LL06b, LL07a, LL07b]. This class of problems attempts to understand the limiting behavior of systems involving a very large number of identical rational agents whose interactions are described by differential games. Mean-field games arise in number of applications including social network dynamics [GMS13], growth theory in economics [LLG10a, ML11], environmental policy [ALT, LL07b], price formation [LL07a], non-renewable resources, oil production, and sustainable development models [LLG10b]. An important class of mean-field games concerns problems with a very large number of agents that can switch between a finite number

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of states, $I_d = \{1, 2, 3, ..., d\}$. Each of the players is allowed to switch between the states by controlling the switching rate of a continuous time Markov chain. These problems have attracted the attention of various researchers, see [GMS10, GMS13, Gue11b, Gue11a].

A very natural question in mean-field games problems concerns its long time behavior. This problem was first addressed in discrete time, finite state in [GMS10]. Then it was investigated in continuous time: for finite state in [GMS13] and for continuous state in [CLLP12]. The arguments in those papers rely on certain uniform convexity and monotonicity hypotheses. In this paper we use a different approach namely by means of Γ-converge. The notion of Γ-convergence was introduced by De Giorgi in the 70's and it consists in a functional approximation which respects the minimization process, that is, under mild hypotheses, the infima of a sequence of functionals converge to the infimum of the limit functional. Moreover, the minimization problem associated with the limit functional has solution. In the present work we start by observing that certain discrete state mean-field games are in fact the Euler-Lagrange equation of a suitable functional. Then we use a Γ-convergence argument to establish the convergence to a stationary mean-field game. We would like to stress that to the best of our knowledge this is the first work where Γ-convergence techniques are used within mean-field games. This method allow us to extend the results in [GMS13] as it requires weaker hypothesis. Furthermore, we believe that a wider class of mean-field games may be tackled by similar methods, in particular it may be possible to address a class of continuous state problems.

This paper is organized as follows: in Section 2 we describe the set-up of finite state mean-field games. The mean-field equations are either, in the time-dependent case, a system of ODEs (2.4), together with initial-terminal data (2.5), or, in the stationary case, by the algebraic equations (2.6). Then, in Section 3, we consider the class of potential mean-field games, which are mean-field games that have a variational formulation. More precisely, under condition (3.1), the mean-field equations (2.4) and (2.5) can be seen as the Euler-Lagrange equation of the functional (3.6). A number of estimates for discrete state mean-field games are recalled in Proposition 4.3, in Section 4, following [GMS13]. We then reformulate, in Section 5, by scaling, the mean-field equations in a form that is particularly convenient for the use of Γ-convergence methods, namely equations (5.4), together with the scaled variational principle (5.12). Several results on Γ-convergence are recalled in Section 6. The main convergence result is then proved in Section 7. We show, in Theorem 7.1, that the functional (5.12), Γ-converges to (5.13). In Corollary 7.2 we establish convergence of the associated infima and minimizers. We end Section 7 by relating the limit minimization problem with the stationary one. In particular, we provide conditions under which a sequence of solutions of (5.1)–(5.2) converge to a stationary solution of (2.6). This results holds under more general conditions than the ones in [GMS13] since we do not require uniform convexity or uniform monotonicity, just strict convexity and monotonicity. Also, we believe that our techniques may extend to continuous models, and therefore it may be possible to address long-time convergence for a different class of models and conditions than the ones considered in [CLLP12].

2 Finite state mean-field games

In this section we review some of the results from [GMS13] concerning finite state mean-field games. For convenience, we will use the same notation and conventions. We consider a (infinite) population of identical agents where each one of them has a state in I_d. The states evolve randomly in time by following a controlled continuous time Markov chain. Each player controls its switching rate in order to optimize a certain functional. The distribution of the players among the different states is given by a probability vector $\theta \in \mathcal{P}(I_d)$, where $\mathcal{P}(I_d)$ is the probability simplex

$$\begin{cases} \theta^1 + ... + \theta^d = 1\,, \\ \theta^i \geq 0 \quad \forall i \in I_d. \end{cases}$$

We fix a reference player and denote its state at time t by the random variable $\mathbf{i}_t$. The process $\mathbf{i}_t$ is a continuous time Markov chain whose switching rate from state $\mathbf{i}_t$ to a state j is denoted by $\alpha_j(t)$. The only available information to this reference player about the other player's state is the probability θ.

The objective of each player is to minimize a certain functional, which is the same for every player since all players are identical. This objective functional is composed of two terms, a running cost and a terminal cost. The running cost of the reference player is determined by a function $c : I_d \times \mathcal{P}(I_d) \times (\mathbb{R}_0^+)^d \to \mathbb{R}$. This running cost $c(i, \theta, \alpha)$ depends on the state i of the player, the distribution of the population among states θ, and the transition rate α_j the player uses to change from state i to state j. The reference player has a terminal cost $\psi : I_d \to \mathbb{R}$. Note that a function $\psi : I_d \to \mathbb{R}$ is identified naturally with a vector in $\mathbb{R}^d$, however in this setting is more natural to consider ψ as a scalar function of a finite set. Given the distribution of the other players for all times, the objective of the reference player is to minimize

$$v^i(t) := \inf_{\alpha} \mathbb{E}^{\alpha}_{\mathbf{i}_t = i} \left[\int_t^T c(\mathbf{i}_s, \theta(s), \alpha(s))\, \mathrm{d}s + \psi^{\mathbf{i}_T} \right], \tag{2.1}$$

where the infimum is taken over all piecewise continuous, progressively measurable controls $\alpha : [t, T] \to (\mathbb{R}_0^+)^d$ and $\mathbb{E}^{\alpha}_{\mathbf{i}_t = i}$ is the expectation conditioned to the event $\mathbf{i}_t = i$ given (the transition rate) α.

We suppose that c is a continuous function and that the map $\alpha \mapsto c(i, \theta, \alpha)$ is convex and does not depend on α_i. We define the generalized Legendre transform of the function $c(i, \theta, \cdot)$, as

$$\begin{aligned} h(z, \theta, i) \;&:=\; \min_{\mu \in (\mathbb{R}_0^+)^d} \left\{ c(i, \theta, \mu) + \sum_{j=1}^d \mu_j (z^j - z^i) \right\} \\ &= \min_{\mu \in (\mathbb{R}_0^+)^d} \left\{ c(i, \theta, \mu) + \mu \cdot \Delta_i z \right\}, \end{aligned} \tag{2.2}$$

where for each $i \in I_d$, $\Delta_i : \mathbb{R}^d \to \mathbb{R}^d$ denotes the difference operator with respect to i, that is,

$$\Delta_i z := (z^1 - z^i, \cdots, z^d - z^i), \quad \text{for all } z = (z^1, \ldots, z^d) \in \mathbb{R}^d. \tag{2.3}$$

We note that $z \mapsto h(z, \theta, i)$ is a concave function.

We suppose that h is locally Lipschitz and differentiable in z. As shown in [GMS13], any solution to

$$-\dot{v}^i = h(\Delta_i v, \theta, i),$$

satisfying $v(T) = \psi$ is the value function for (2.1), where $\dot{v}$ stands for $\mathrm{d}v/\mathrm{d}t$. Additionally, if we define

$$\alpha_j^*(\Delta_i z, \theta, i) := \frac{\partial}{\partial z^j}\big(h(\Delta_i z, \theta, i)\big),$$

the feedback strategy $\alpha_i^*(\Delta_j v, \theta, j)$ is an optimal switching policy for a player in state i. The mean-field Nash equilibrium hypothesis assumes that every player uses this switching strategy. This leads to the following system

$$\begin{cases} \dot{\theta}^i = \displaystyle\sum_{j=1}^{d} \theta^j \alpha_i^*(\Delta_j v, \theta, j), \\ -\dot{v}^i = h(\Delta_i v, \theta, i), \end{cases} \tag{2.4}$$

together with the initial-terminal conditions

$$\theta(0) = \theta_0, \qquad v^i(T) = \psi^i, \tag{2.5}$$

where θ_0 is the initial distribution of players.

In addition to the time dependent problem, we will also need to consider stationary solutions, as defined next:

Definition 2.1. *A triplet $(\bar{\theta}, \bar{v}, \bar{\lambda}) \in \mathcal{P}(I_d) \times \mathbb{R}^d \times \mathbb{R}$ is called a stationary solution of* (2.4) *if*

$$\begin{cases} \displaystyle\sum_{j=1}^{d} \bar{\theta}^j \alpha_i^*(\Delta_j \bar{v}, \bar{\theta}, j) = 0\,, \\ h(\Delta_i \bar{v}, \bar{\theta}, i) = \bar{\lambda}\,, \end{cases} \tag{2.6}$$

for all $i \in I_d$.

If $(\bar{\theta}, \bar{v}, \bar{\lambda})$ is a stationary solution for the MFG equations, then $(\bar{\theta}, \bar{v} - \bar{\lambda} t \mathbf{1})$, where $\mathbf{1} := (1, ..., 1)$, solves (2.4).

In [GMS13] the convergence for the solutions to the time-dependent problem to stationary solutions was studied using strong convexity and monotonicity hypotheses.

3 Potential mean-field games

An important class of examples that we consider in this paper are potential mean-field games as discussed in [GMS13] and [Gue11b]. In these mean-field games, (2.4) can be regarded as an Euler-Lagrange equation of a suitable functional. For continuous state, the variational principles discussed in this section are the analog to the results in [GPSM12, GSM11].

Suppose h has the form

$$h(z, \theta, i) = \tilde{h}(z, i) + f(\theta, i), \tag{3.1}$$

where $\tilde{h} : \mathbb{R}^d \times I_d \to \mathbb{R}$ and $f : \mathbb{R}^d \times I_d \to \mathbb{R}$ is the gradient of a convex function. More precisely, we suppose that there exists a convex function $F : \mathbb{R}^d \to \mathbb{R}$ such that $\frac{\partial F}{\partial \theta_i} = f(\theta, i)$.

Let $H : \mathbb{R}^{2d} \to \mathbb{R}$ be given by

$$
\begin{aligned}
H(v, \theta) :=& \sum_{i=1}^{d} \theta^i \tilde{h}(\Delta_i v, i) + F(\theta) \\
=& \theta \cdot \tilde{h}(\Delta . v, \cdot) + F(\theta) \, ,
\end{aligned}
\tag{3.2}
$$

where $\tilde{h}(\Delta . z, \cdot)$, with $z \in \mathbb{R}^d$, represents the vector in $\mathbb{R}^d$ whose i^{th} coordinate is $\tilde{h}(\Delta_i z, i)$. A direct computation shows that (2.4) can be written as

$$
\begin{cases}
\dfrac{\partial H}{\partial v^j} = \dot{\theta}^j \, , \\[2ex]
\dfrac{\partial H}{\partial \theta^j} = -\dot{v}^j .
\end{cases}
\tag{3.3}
$$

This means the flow generated by equation (2.4) is Hamiltonian.

If the function F is strictly convex in θ then the Hamiltonian H is strictly convex in θ. This allow us to consider the Legendre transform

$$
\begin{aligned}
L(v, \dot{v}) :=& \sup_{\theta \in \mathbb{R}^d} \big\{ - \dot{v} \cdot \theta - H(v, \theta) \big\} \\
=& \sup_{\theta \in \mathbb{R}^d} \big\{ - (\dot{v} + \tilde{h}(\Delta . v, \cdot)) \cdot \theta - F(\theta) \big\} = F^*(\dot{v} + \tilde{h}(\Delta . v, \cdot)).
\end{aligned}
\tag{3.4}
$$

Moreover, assuming, in addition, that F has superlinear growth at infinity, then by the properties of the Legendre transform, given a solution (υ, ϑ) of (2.4) satisfying (2.5) it holds

$$
\vartheta(t) = -\nabla F^*(\dot{\upsilon}(t) + \tilde{h}(\Delta . \upsilon(t), \cdot)). \tag{3.5}
$$

In particular,

$$
\theta_0 = -\nabla F^*(\dot{\upsilon}(0) + \tilde{h}(\Delta . \upsilon(0), \cdot)).
$$

From this we conclude that any such υ is a critical point of the functional

$$
\int_0^T F^*(\dot{v} + \tilde{h}(\Delta . v, \cdot)) \, \mathrm{d}t - \theta_0 \cdot v(0), \tag{3.6}
$$

where we are looking for critical points v that have fixed boundary condition at T, namely $v(T) = \psi$.

Conversely, let υ be a critical point of the functional (3.6) and let ϑ be given by (3.4) with v replaced by υ so that (3.5) holds. Then (υ, ϑ) satisfies (2.4) and (2.5).

In order for every component of ϑ to be non-negative, we require F^* to be non-increasing in each coordinate, that is,

$$
\frac{\partial F^*}{\partial p_j}(p) \leq 0,
$$

for all $p \in \mathbb{R}^d$ and $j \in I_d$. From the Euler-Lagrange equation we have that

$$\sum_{i=1}^{d} \dot{\vartheta}_i = 0,$$

thus $\vartheta(t)$ is a probability vector for all t.

In the stationary setting, for every fixed $\lambda \in \mathbb{R}$, we consider simply the problem of minimizing

$$F^*(\tilde{h}(\Delta.v, \cdot) - \lambda \mathbf{1}) - \lambda\,, \tag{3.7}$$

where the minimization is performed over all $v \in \mathbb{R}^d$ satisfying $\sum_{i=1}^d v^i = 0$, and, we recall, $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^d$. If $\bar{v}$ is a critical point of this problem, then setting

$$\bar{\theta}^j := -\frac{\partial F^*}{\partial p_j}(\tilde{h}(\Delta.\bar{v}, \cdot) - \lambda \mathbf{1}), \quad j \in I_d,$$

we conclude that $(\bar{\theta}, \bar{v}, \lambda)$ satisfies (2.6). In particular, if $\bar{\lambda} \in \mathbb{R}$ is such that

$$-\sum_{j=1}^{d} \frac{\partial F^*}{\partial p_j}(\tilde{h}(\Delta.v, \cdot) - \bar{\lambda} \mathbf{1}) = 1, \tag{3.8}$$

then $(\bar{\theta}, \bar{v}, \bar{\lambda})$ is a stationary solution in the sense of Definition 2.1. We further observe that such $\bar{\lambda}$ exists: if we minimize (3.7) over $\lambda \in \mathbb{R}$, then denoting by $\bar{\lambda}$ a corresponding critical point we conclude that (3.8) holds.

We will need also the following observation. For fixed $\lambda \in \mathbb{R}$, consider the problem of minimizing

$$\int_0^1 F^*(\tilde{h}(\Delta.v, \cdot) - \lambda \mathbf{1})\, \mathrm{d}t - \lambda \tag{3.9}$$

among all continuous functions $v : [0, 1] \to \mathbb{R}^d$ satisfying $\sum_{i=1}^d v_i(t) = 0$. By a simple application of the Jensen's inequality, taking into account that F^* is a convex, componentwise non-increasing function, and $\tilde{h}(\cdot, i)$, $i \in I_d$, is a concave function, it is possible to show that it suffices to consider minimizers to (3.9) in the class of constant functions v. Therefore it is enough to look at minimizers of (3.7).

4 Estimates for finite state mean-field games

We recall now a number of estimates from [GMS13] concerning finite state mean-field games. We start with two definitions:

Definition 4.1. *Let $v \in \mathbb{R}^d$. In $\mathbb{R}^d/\mathbb{R}$ we define the norm*

$$\|v\|_\sharp := \inf_{\lambda \in \mathbb{R}} |v + \lambda \mathbf{1}|_\infty.$$

It can be checked that for all $v \in \mathbb{R}^d$,

$$\|v\|_\sharp = \frac{\max_{i \in I_d} v^i - \min_{i \in I_d} v^i}{2}.$$

Definition 4.2. *Let $\langle u\rangle = \frac{1}{d}\sum_{j=1}^{d} u^j$. We say that $h : \mathbb{R}^d \times \mathcal{P}(I_d) \times I_d \to \mathbb{R}$ is contractive if there exists $M > 0$ such that if $\|v\|_\sharp > M$, then the two following conditions hold for all $\theta \in \mathcal{P}(I_d)$ and $i \in I_d$:*

$$(\Delta_i v)^j \leq 0 \text{ for all } j \in I_d \text{ implies } h(\Delta_i v, \theta, i) - \langle h(v, \theta, \cdot)\rangle < 0\,, \tag{4.1}$$
$$(\Delta_i v)^j \geq 0 \text{ for all } j \in I_d \text{ implies } h(\Delta_i v, \theta, i) - \langle h(v, \theta, \cdot)\rangle > 0\,. \tag{4.2}$$

Many mean-field games are contractive. For instance, in [GMS13] the authors prove that to the running cost

$$c(i, \theta, \alpha) = \sum_{j=1}^{d} \frac{\alpha_j^2}{2} + f(\theta, i)\,, \tag{4.3}$$

for $f(\theta, i)$ continuous in $\theta \in \mathcal{P}(I_d)$, corresponds a contractive Hamiltonian:

$$h(\Delta_i v, \theta, i) = f(\theta, i) - \frac{1}{2}\sum_{j=1}^{d}[(u^i - u^j)^+]^2\,. \tag{4.4}$$

For contractive mean-field games the following result was established in [GMS13]:

Proposition 4.3. *Suppose $h : \mathbb{R}^d \times \mathcal{P}(I_d) \times I_d \to \mathbb{R}$ given by (2.2) is contractive. Then*

(a) For M large enough, the set $\left\{v \in \mathbb{R}^d, \|v\|_\sharp < M\right\} \times \mathcal{P}(I_d)$ is invariant backwards in time by the flow of equation (2.4).

(b) There exist a stationary solution of (2.4).

5 Scaling

In order to study the long time behaviour of mean-field games we introduce a scaled version of (2.4), where $\epsilon = \frac{1}{T}$,

$$\begin{cases} \epsilon\dot{\theta}_\epsilon^i = \displaystyle\sum_{j=1}^{d} \theta_\epsilon^j \alpha_i^*(\Delta_j v_\epsilon, \theta_\epsilon, j), \\ -\epsilon\dot{v}_\epsilon^i = h(\Delta_i v_\epsilon, \theta_\epsilon, i), \end{cases} \tag{5.1}$$

together with the initial-terminal conditions

$$\theta_\epsilon(0) = \theta_0, \qquad v_\epsilon^i(1) = \psi^i. \tag{5.2}$$

We can assume $\sum_{i=1}^{d} \psi^i = 0$, without loss of generality. We observe also that the scaling in time does not change the bounds in Proposition 4.3. Hence, if $(v_\epsilon, \theta_\epsilon)$ solves (5.1)–(5.2) with h as in Proposition 4.3, then (see [GMS13])

$$\sup_{t\in[0,1]} \sup_{\epsilon>0} \|v_\epsilon(t)\|_\sharp < +\infty. \tag{5.3}$$

Assume that h is as in Proposition 4.3. In order to write the scaled version of the functional (3.6) associated with (5.1)–(5.2) for potential mean-field games as in Section 3, in a convenient form for the use of Γ-convergence we decompose v_ϵ as follows. Let $\lambda_\epsilon \in \mathbb{R}$, $u_\epsilon : [0,1] \to \mathbb{R}^d$, and $w_\epsilon : [0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned}
\lambda_\epsilon &:= \int_0^1 \sum_{i=1}^d h(\Delta_i v_\epsilon, \theta_\epsilon, i)\,\mathrm{d}t\,,\\
w_\epsilon(t) &:= \frac{\epsilon}{d}\sum_{i=1}^d v_\epsilon^i(t) - \lambda_\epsilon(1-t)\,,\\
u_\epsilon^i(t) &:= v_\epsilon^i(t) - \frac{1}{\epsilon}w_\epsilon(t) - \frac{\lambda_\epsilon}{\epsilon}(1-t).
\end{aligned}$$

Observing that $\Delta_i u_\epsilon = \Delta_i v_\epsilon$ for all $i \in I_d$, (5.1) becomes

$$\begin{cases} \epsilon\dot\theta_\epsilon^i = \displaystyle\sum_{j=1}^d \theta_\epsilon^j \alpha_i^*(\Delta_j u_\epsilon, \theta_\epsilon, j),\\ \lambda_\epsilon - \dot w_\epsilon - \epsilon \dot u_\epsilon^i = h(\Delta_i u_\epsilon, \theta_\epsilon, i). \end{cases} \tag{5.4}$$

We claim that

$$\sup_{\epsilon>0}|\lambda_\epsilon| < +\infty, \qquad \sup_{t\in[0,1]}\sup_{\epsilon>0}\|u_\epsilon(t)\|_\sharp < +\infty, \tag{5.5}$$

$$\sum_{i=1}^d u_\epsilon^i(t) = 0 \text{ for all } t\in[0,1] \text{ and } \epsilon>0, \tag{5.6}$$

$$u_\epsilon^i(1) = \psi^i \text{ for all } i \in I^d \text{ and } \epsilon > 0, \tag{5.7}$$

$$\sup_{\epsilon>0}\|\epsilon\dot u_\epsilon\|_\infty < +\infty, \tag{5.8}$$

$$w_\epsilon(0) = 0,\ \ w_\epsilon(1) = 0,\ \text{ for all } \epsilon > 0, \tag{5.9}$$

$$\sup_{\epsilon>0}\|\dot w_\epsilon\|_\infty < +\infty. \tag{5.10}$$

In fact, (5.5) is a result of the hypotheses on h, (5.3), and the equality $\|v_\epsilon\|_\sharp = \|u_\epsilon\|_\sharp$. Condition (5.6) follows from the definition of u_ϵ and w_ϵ. On the other hand, since $\sum_{i=1}^d v_\epsilon^i(1) = \sum_{i=1}^d \psi^i = 0$, we deduce the second condition in (5.9), which in turn yields (5.7) in view of (5.2). We now notice that

$$\dot w_\epsilon = \frac{\epsilon}{d}\sum_{i=1}^d \dot v_\epsilon^i(t) + \lambda_\epsilon = -\frac{1}{d}\sum_{i=1}^d h(\Delta_i v_\epsilon, \theta_\epsilon, i) + \lambda_\epsilon\,, \tag{5.11}$$

from which, together with (5.3), the first estimate in (5.5), and the hypotheses on h, we obtain (5.10). Moreover, integrating (5.11) over $[0,1]$ and using the equality $w_\epsilon(1) = 0$ already proved and the definition of λ_ϵ, we get $w_\epsilon(0) = 0$. Thus, (5.9) holds. Finally, (5.8) follows from the identity $\epsilon\dot u_\epsilon^i = -\epsilon h(\Delta_i v_\epsilon, \theta_\epsilon, i) - \dot w_\epsilon^i + \lambda_\epsilon$ having in mind the uniform bounds established above. Thus (5.5)–(5.10) hold.

We now observe that the system of equations (5.4), together with (5.5)–(5.10), suggests that in the limit $\epsilon \to 0$ we have $w_\epsilon \to 0$ and $(\theta_\epsilon, u_\epsilon, \lambda_\epsilon) \to (\bar\theta, \bar u, \bar\lambda)$, where $(\bar\theta, \bar u, \bar\lambda)$ solves (2.6).

From the variational point of view and for potential mean-field games as in Section 3 (see (3.6)), observing that $\theta_0 \cdot \lambda\mathbf{1} = \lambda$ since $\theta_0 \in \mathcal{P}(I_d)$, we look for minimizers of

$$\int_0^1 F^*(\dot{w}\mathbf{1} + \epsilon\dot{u} + \tilde{h}(\Delta.u, \cdot) - \lambda\mathbf{1})\,\mathrm{d}t - \epsilon\theta_0 \cdot u(0) - \lambda \tag{5.12}$$

over $\lambda \in \mathbb{R}$, $u : [0,1] \to \mathbb{R}^d$, and $w : [0,1] \to \mathbb{R}$ according to (5.5)–(5.10).

At least formally, the limit of the functional (5.12) is

$$\int_0^1 F^*(\dot{w}\mathbf{1} + \tilde{h}(\Delta.u, \cdot) - \lambda\mathbf{1})\,\mathrm{d}t - \lambda, \tag{5.13}$$

which corresponds to (3.9) provided that w does not depend on t. In particular the boundary conditions are lost in the limiting procedure. To justify rigorously this limiting procedure we need to use Γ-convergence techniques, which we now proceed to address.

6 Preliminaries on Γ-convergence

In this section we recall some standard results on sequential lower semicontinuity of certain funtionals and on Γ-convergence. We refer the reader to [DM93, Bra02] for a comprehensive treatment and bibliography on Γ-convergence.

Theorem 6.1. *([FL07, Thm. 5.14]) Let B be a Borel subset of $\mathbb{R}^N$ with finite measure, let $1 \leqslant p \leqslant +\infty$, and let $f : \mathbb{R}^d \to (-\infty, +\infty]$ be a lower semicontinuous function. Assume that there exists $C > 0$ such that*

$$f(z) \geqslant -C(1 + |z|^p) \text{ for all } z \in \mathbb{R}^d \text{ if } 1 \leqslant p < +\infty,$$
$$f \text{ is locally bounded from below if } p = +\infty.$$

Then the functional

$$u \in L^p(B; \mathbb{R}^d) \mapsto \int_B f(u(x))\,\mathrm{d}x$$

is sequentially lower semicontinuous with respect to the weak convergence in $L^p(B; \mathbb{R}^d)$ (weak star if $p = +\infty$) if, and only if, f is convex.

Definition 6.2. *Let X be a Banach space, let $F : X \to \overline{\mathbb{R}}$, and let $\delta > 0$. We say that $x \in X$ is a δ-minimizer of F in X if*

$$F(x) \leqslant \max\Big\{ \inf_{y \in X} F(y) + \delta, -\frac{1}{\delta} \Big\}.$$

Remark 6.3. If $\inf_{y\in X} F(y) > -\infty$ and if δ is small enough, then x is a δ-minimizer of F in X if, and only if, $F(x) \leqslant \inf_{y\in X} F(y) + \delta$.

Theorem 6.4. *([DM93, Prop. 8.16, Thm. 7.8, and Cor. 7.20]) Let X be a reflexive Banach space endowed with its weak topology, and let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a sequence of functionals $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ equi-coercive in the weak topology of X. Assume that there is a functional $\mathcal{F} : X \to \overline{\mathbb{R}}$ satisfying the two following conditions:*

i) for every $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly converging to x in X, one has $\mathcal{F}(x) \leqslant \liminf_{n\to+\infty} \mathcal{F}_n(x_n)$;

ii) for every $x \in X$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly converging to x in X such that $\mathcal{F}(x) = \lim_{n\to+\infty} \mathcal{F}_n(x_n)$.

Then

$$\min_{x\in X} \mathcal{F}(x) = \lim_{n\to+\infty} \inf_{x\in X} \mathcal{F}_n(x).$$

Moreover, if for each $n \in \mathbb{N}$ x_n is a minimizer of $\mathcal{F}_n$ in X (or, more generally, a δ_n-minimizer, where $\{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0) and x is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$, then x is a minimizer of $\mathcal{F}$ in X, and

$$\mathcal{F}(x) = \limsup_{n\to+\infty} \mathcal{F}_n(x_n).$$

If $\{x_n\}_{n\in\mathbb{N}}$ weakly converges to x in X, then x is a minimizer of $\mathcal{F}$ in X, and

$$\mathcal{F}(x) = \lim_{n\to+\infty} \mathcal{F}_n(x_n).$$

Remark 6.5. In view of *i)*, condition *ii)* in Theorem 6.4 may be replaced by

ii)' for every $x \in X$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly converging to x in X such that $\mathcal{F}(x) \geqslant \limsup_{n\to+\infty} \mathcal{F}_n(x_n)$.

In the language of Γ-convergence, if $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\mathcal{F}$ are as in Theorem 6.4, then $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is said to Γ-converge to F in X as $n \to +\infty$ with respect to the weak convergence in X. Condition *i)* is called the "liminf inequality", condition *ii)'* the "limsup inequality", while the sequence in condition *ii)* (or, equivalently, in *ii)'*) is called "a recovery sequence".

7 Convergence of functionals, and its minima, associated with mean-field games

In this section we study the asymptotic behavior as $\epsilon \to 0$ of the functionals in (5.12) subjected to the conditions (5.5)–(5.10). The space of continuous functions is not the most appropriate one for this study as it is not a reflexive Banach space. For this reason we extend the functionals to the product space $L^p \times W_0^{1,p}$, for $p \in (1, +\infty)$, in a natural way (see (7.2) below). In Theorem 7.1 we establish the Γ-convergence of the sequence of these functionals and in Corollary 7.2 the convergence of the associated infima and minimizers. These two results provide a rigorous proof of the heuristics in the end of Section 5. We finish Section 7 with Remark 7.3, which relates the limit minimization problem obtained in Corollary 7.2 with the stationary setting (3.7) and (2.6).

We start by making precise the hypotheses under which the results in this section hold.

We suppose that $F^* : \mathbb{R}^d \to \mathbb{R}$ is a non-incresing, convex function, that is, F^* is convex and $F^*(z) \geqslant F^*(w)$ whenever $z = (z^1, \ldots, z^d)$ and $w = (w^1, \ldots, w^d) \in \mathbb{R}^d$ are such that $z^i \leqslant w^i$ for all $i \in I_d = \{1, \cdots, d\}$. We assume further that $\tilde{h} : \mathbb{R}^d \times I_d \to \mathbb{R}$ is a locally Lipschitz function in the first variable and such that $\tilde{h}(\Delta_i \cdot, i)$ is a concave function for all $i \in I_d$.

Recalling that $\Delta_i : \mathbb{R}^d \to \mathbb{R}^d$ denotes the difference operator with respect to $i \in I_d$ (see (2.3)), we observe that for all $c > 0$ there is a constant $L > 0$, only depending on c and d, such that

$$|\tilde{h}(\Delta_i z, i) - \tilde{h}(\Delta_i w, i)| \leqslant L|z - w| \tag{7.1}$$

for all $i \in I_d$, whenever $z, w \in \mathbb{R}^d$ are such that $|\Delta_j z|, |\Delta_j w| \leqslant c$ for all $j \in I_d$. In fact, it suffices to notice that

$$\begin{aligned}
|\Delta_i z - \Delta_i w| &= \left(\sum_{j=1}^{d} |(z^j - w^j) - (z^i - w^i)|^2 \right)^{1/2} \\
&\leqslant \left(\sum_{\substack{j=1 \\ j \neq i}}^{d} 2|(z^j - w^j)|^2 + 2(d-1)|z^i - w^i|^2 \right)^{1/2} \\
&\leqslant \sqrt{2(d-1)}|z - w|.
\end{aligned}$$

Finally, let $\tilde{\mathbf{h}} : \mathbb{R}^d \to \mathbb{R}^d$ be the function defined by

$$\tilde{\mathbf{h}}(z) := (\tilde{h}^1(z), \cdots, \tilde{h}^d(z)), \quad \text{with } \tilde{h}^i(z) := \tilde{h}(\Delta_i z, i) \text{ for all } i \in I_d,$$

and let R_0, M_0, and $\bar{M}_0$ be (arbitrary) positive constants. For each $\epsilon > 0$ we define the functional $\mathcal{F}_\epsilon : L^p((0,1); \mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ by setting

$$\mathcal{F}_\epsilon(u, w, \lambda) := \begin{cases} \displaystyle\int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon\dot{u}(t) + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\,\mathrm{d}t - \epsilon\theta_0 \cdot u(0) - \lambda \\ \qquad \text{if } u \in \mathcal{A}_\psi,\ \|u(\cdot)\|_\sharp \leqslant \bar{M}_0,\ \displaystyle\sum_{i=1}^d u^i(\cdot) = 0,\ \mathcal{L}^1\text{-a.e. in } (0,1), \\ \qquad \max\left\{ \displaystyle\int_0^1 |\epsilon\dot{u}(t)|^p\,\mathrm{d}t, \|\dot{w}\|_{L^\infty(0,1)} \right\} \leqslant M_0,\ |\lambda| \leqslant R_0, \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{7.2}$$

where $\psi \in \mathbb{R}^d$ is such that $\sum_{i=1}^d \psi^i = 0$ and $\|\psi\|_\sharp \leqslant \bar{M}_0$,

$$\mathcal{A}_\psi := \big\{ u \in W^{1,p}((0,1); \mathbb{R}^d) \colon\ u(1) = \psi \big\},$$

and, we recall, $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^d$ and $\|z\|_\sharp = \frac{1}{2}\big(\max_{i \in I_d} z^i - \min_{i \in I_d} z^i\big)$ for $z = (z^1, \ldots, z^d) \in \mathbb{R}^d$.

Let $\mathcal{F}_0 : L^p((0,1); \mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ be the functional defined by

$$\mathcal{F}_0(u, w) := \begin{cases} \displaystyle\int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\,\mathrm{d}t - \lambda \\ \qquad \text{if } \|u(\cdot)\|_\sharp \leqslant \bar{M}_0,\ \displaystyle\sum_{i=1}^d u^i(\cdot) = 0,\ \mathcal{L}^1\text{-a.e. in } (0,1), \\ \qquad \|\dot{w}\|_{L^\infty(0,1)} \leqslant M_0,\ |\lambda| \leqslant R_0, \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{7.3}$$

The following Γ-convergence's result holds.

Theorem 7.1. *Let $\mathcal{F}_\epsilon$, $\epsilon > 0$, and $\mathcal{F}_0$ be the functionals given by* (7.2) *and* (7.3), *respectively. Then the sequence $\{\mathcal{F}_\epsilon\}_{\epsilon>0}$ Γ-converges as $\epsilon \to 0^+$ to $\mathcal{F}_0$ with respect to the weak convergence in $L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$.*

Proof. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero. We will proceed in two steps.

Step 1. We prove that for all $\{(u_n, w_n, \lambda_n)\}_{n\in\mathbb{N}} \subset L^p((0,1);\mathbb{R}^d)\times W_0^{1,p}(0,1)\times \mathbb{R}$ and $(u, w, \lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$ such that $u_n \rightharpoonup u$ weakly in $L^p((0,1);\mathbb{R}^d)$, $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(0,1)$, and $\lambda_n \to \lambda$ in $\mathbb{R}$ as $n \to +\infty$, the following inequality holds

$$\mathcal{F}_0(u, w, \lambda) \leqslant \liminf_{n\to+\infty} \mathcal{F}_{\epsilon_n}(u_n, w_n, \lambda_n). \tag{7.4}$$

To prove (7.4), we may assume without loss of generality that

$$\liminf_{n\to+\infty} \mathcal{F}_{\epsilon_n}(u_n, w_n, \lambda_n) = M < +\infty.$$

Then for all $n \in \mathbb{N}$, $u_n \in \mathcal{A}_\psi$, $\|u_n(\cdot)\|_\sharp \leqslant \bar{M}_0$ and $\sum_{i=1}^d u_n^i(\cdot) = 0$ $\mathcal{L}^1$-a.e. in $(0,1)$, $\int_0^1 |\epsilon_n \dot{u}_n(t)|^p \,\mathrm{d}t \leqslant M_0$, $\|\dot{w}_n\|_{L^\infty(0,1)} \leqslant M_0$, and $|\lambda_n| \leqslant R_0$. Moreover, extracting a subsequence (that we will not relabel), we may assume that

$$\begin{aligned} M &= \lim_{n\to+\infty} \mathcal{F}_{\epsilon_n}(u_n, w_n, \lambda_n) \\ &= \lim_{n\to+\infty} \bigg(\int_0^1 F^*\big(\dot{w}_n(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda_n \mathbf{1}\big)\,\mathrm{d}t \\ &\qquad\qquad -\epsilon_n \theta_0 \cdot u_n(0) - \lambda_n \bigg), \end{aligned} \tag{7.5}$$

$$\epsilon_n\, u_n \rightharpoonup 0 \ \text{ weakly in } W^{1,p}((0,1);\mathbb{R}^d) \text{ as } n \to +\infty, \tag{7.6}$$

$$w_n \overset{\star}{\rightharpoonup} w \ \text{ weakly star in } W^{1,\infty}(0,1) \text{ as } n \to +\infty. \tag{7.7}$$

We claim that $\|u(\cdot)\|_\sharp \leqslant \bar{M}_0$ and $\sum_{i=1}^d u^i(\cdot) = 0$ $\mathcal{L}^1$-a.e. in $(0,1)$, $\|\dot{w}\|_{L^\infty(0,1)} \leqslant M_0$, and $|\lambda| \leqslant R_0$. In fact, for all $n \in \mathbb{N}$, and for $\mathcal{L}^1$-a.e. $t \in (0,1)$,

$$\bar{M}_0 \geqslant \|u_n(t)\|_\sharp = \frac{\max_{i\in I_d} u_n^i(t) - \min_{i\in I_d} u_n^i(t)}{2} \geqslant \frac{u_n^j(t) - u_n^k(t)}{2} \tag{7.8}$$

for all $j, k \in I_d$. Let $t_0 \in (0,1)$ be a Lebesgue point for u and let $\delta > 0$. Using the weak convergence $u_n \rightharpoonup u$ in $L^p((0,1);\mathbb{R}^d)$ as $n \to +\infty$, we conclude from (7.8) that

$$\bar{M}_0 \geqslant \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \frac{u^j(t) - u^k(t)}{2}\,\mathrm{d}t$$

for all $j, k \in I_d$. Thus, letting $\delta \to 0^+$,

$$\bar{M}_0 \geqslant \frac{u^j(t_0) - u^k(t_0)}{2}\cdot \tag{7.9}$$

Taking the maximum over $j \in I_d$ and then the maximum over $k \in I_d$ in (7.9), we conclude that

$$\|u(t)\|_\sharp \leqslant \bar{M}_0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1).$$

On the other hand, by Theorem 6.1 applied to the real-valued convex function $z \in \mathbb{R}^d \mapsto f(z) := \big|\sum_{i=1}^d z^i\big|^p$ we get

$$\int_0^1 \Big|\sum_{i=1}^d u^i(t)\Big|^p \,\mathrm{d}t \leqslant \liminf_{n\to+\infty} \int_0^1 \Big|\sum_{i=1}^d u_n^i(t)\Big|^p \,\mathrm{d}t = 0.$$

Thus, $\sum_{i=1}^d u_n^i(\cdot) = 0$ $\mathcal{L}^1$-a.e. in $(0,1)$. We now observe that in view of the lower semicontinuity of the L^∞-norm with respect to the weak star convergence in L^∞,

$$\|\dot{w}\|_{L^\infty(0,1)} \leqslant \liminf_{n\to+\infty} \|\dot{w}_n\|_{L^\infty(0,1)} \leqslant M_0.$$

Finally, $|\lambda| \leqslant R_0$ since $|\lambda_n| \leqslant R_0$ and $\lambda_n \to \lambda$ in $\mathbb{R}$. Hence the claim holds, which in particular implies that

$$\mathcal{F}_0(u,w,\lambda) = \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\,\mathrm{d}t - \lambda.$$

We now prove that (up to a not relabeled subsequence)

$$\dot{w}_n(\cdot)\mathbf{1} + \epsilon_n \dot{u}_n(\cdot) + \tilde{\mathbf{h}}(u_n(\cdot)) - \lambda_n\mathbf{1} \rightharpoonup \dot{w}(\cdot)\mathbf{1} + \eta(\cdot) - \lambda\mathbf{1} \tag{7.10}$$

weakly in $L^p((0,1);\mathbb{R}^d)$ as $n \to +\infty$, for some $\eta \in L^p((0,1);\mathbb{R}^d)$.

In view of (7.8), we deduce that for all $j,k \in I_d$, $|u_n^j(t) - u_n^k(t)| \leqslant 2\bar{M}_0$, and so, $|\Delta_i u_n(t)| \leqslant \sqrt{2d}\bar{M}_0$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. This, together with the Lipschitz condition (7.1), yields the existence of a positive constant L, only depending on $\bar{M}_0$ and d, such that for $\mathcal{L}^1$-a.e. $t \in (0,1)$,

$$|\tilde{\mathbf{h}}(u_n(t))| \leqslant |\tilde{\mathbf{h}}(u_n(t)) - \tilde{\mathbf{h}}(0)| + |\tilde{\mathbf{h}}(0)| \leqslant L|u_n(t)| + |\tilde{\mathbf{h}}(0)|.$$

Hence

$$\sup_{n\in\mathbb{N}} \int_0^1 |\tilde{\mathbf{h}}(u_n(t))|^p \,\mathrm{d}t \leqslant \widetilde{C}$$

for some positive constant $\widetilde{C}$. Therefore, (up to a not relabeled subsequence)

$$\tilde{\mathbf{h}}(u_n(\cdot)) \rightharpoonup \eta(\cdot) \tag{7.11}$$

weakly in $L^p((0,1);\mathbb{R}^d)$ as $n \to +\infty$, for some $\eta \in L^p((0,1);\mathbb{R}^d)$, which, together with convergences $\lambda_n \to \lambda$ in $\mathbb{R}$, $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(0,1)$, and (7.6), proves (7.10). We observe further that due to the continuity of the trace operator, we have that

$$\epsilon_n u_n(0) \to 0 \tag{7.12}$$

in $\mathbb{R}^d$ as $n \to +\infty$.

We now prove that $\mathcal{L}^1$-a.e. in $(0,1)$ and for all $i \in I_d$,

$$\eta^i(\cdot) \leqslant h^i(u(\cdot)). \tag{7.13}$$

In fact, let $t_0 \in (0,1)$ be a Lebesgue point for η and $\tilde{\mathbf{h}}(u)$, and let $\delta > 0$. Being h^i a real-valued concave function, it is, in particular, continuous and thus bounded from above by an affine function (see, e.g., [FL07, Prop. 4.75]). Hence, in view of convergences $u_n \rightharpoonup u$ and $\tilde{\mathbf{h}}(u_n(\cdot)) \rightharpoonup \eta(\cdot)$ weakly in $L^p((0,1);\mathbb{R}^d)$ as $n \to +\infty$, Theorem 6.1 implies that

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} \eta^i(t)\,\mathrm{d}t = \limsup_{n\to+\infty} \frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} h^i(u_n(t))\,\mathrm{d}t \leqslant \frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} h^i(u(t))\,\mathrm{d}t,$$

from which we conclude (7.13) by letting $\delta \to 0^+$.

We finally observe that since F^* is a real-valued convex function, it is bounded from below by an affine function. Therefore, Theorem 6.1 and (7.10) yield

$$\begin{aligned}\int_0^1 & F^*\big(\dot{w}(t)\mathbf{1} + \eta(t) - \lambda\mathbf{1}\big)\,\mathrm{d}t \\ &\leqslant \liminf_{n\to+\infty}\int_0^1 F^*\big(\dot{w}_n(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda_n\mathbf{1}\big)\,\mathrm{d}t,\end{aligned}$$

which, together with the hypothesis that F^* is non-increasing, (7.13), (7.12), and the convergence $\lambda_n \to \lambda$ in $\mathbb{R}$, concludes Step 1.

Step 2. We prove that for all $(u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$ there exists a sequence $\{(u_n,w_n,\lambda_n)\}_{n\in\mathbb{N}} \subset L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$ such that $u_n \rightharpoonup u$ weakly in $L^p((0,1);\mathbb{R}^d)$, $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(0,1)$, and $\lambda_n \to \lambda$ in $\mathbb{R}$ as $n \to +\infty$, and such that

$$\mathcal{F}_0(u,w,\lambda) \geqslant \limsup_{n\to+\infty} \mathcal{F}_{\epsilon_n}(u_n,w_n,\lambda_n). \tag{7.14}$$

Let $(u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$ be given. The only nontrivial case is the case in which $\|u(\cdot)\|_\sharp \leqslant \bar{M}_0$ and $\sum_{i=1}^d u^i(\cdot) = 0$ $\mathcal{L}^1$-a.e. in $(0,1)$, $\|\dot{w}\|_{L^\infty(0,1)} \leqslant M_0$, and $|\lambda| \leqslant R_0$, otherwise it suffices to define $(u_n,w_n,\lambda_n) := (u,w,\lambda)$ for all $n \in \mathbb{N}$. Fix any such (nontrivial) triplet (u,w,λ).

Let $\rho \in C_c^\infty(\mathbb{R})$ be the function defined by

$$\rho(t) := \begin{cases} c\,\mathrm{e}^{-\frac{1}{t^2-1}} & \text{if } t \in (-1,1), \\ 0 & \text{if } t \in \mathbb{R}\backslash(-1,1), \end{cases}$$

where $c > 0$ is such that $\int_{\mathbb{R}} \rho(t)\,\mathrm{d}t = 1$. Let $c_{u,\rho} := \|\rho\|_{C^1(\mathbb{R})}\|u\|_{L^1((0,1);\mathbb{R}^d)} \in \mathbb{R}^+$.

Substep 2.1. We construct a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^p((0,1);\mathbb{R}^d)$ satisfying the following conditions:

$$u_n \to u \text{ in } L^p((0,1);\mathbb{R}^d) \text{ as } n \to +\infty, \tag{7.15}$$

$$\sqrt[4]{\epsilon_n}\,\|u_n\|_{L^\infty((0,1);\mathbb{R}^d)} \leqslant c_{u,\rho}, \tag{7.16}$$

$$\sqrt{\epsilon_n}\,\|\dot{u}_n\|_{L^\infty((0,1);\mathbb{R}^d)} \leqslant c_{u,\rho}, \tag{7.17}$$

$$\sum_{i=1}^d u_n^i(\cdot) = 0 \ \mathcal{L}^1\text{-a.e. in } (0,1), \tag{7.18}$$

$$\|u_n(\cdot)\|_\sharp \leqslant \bar{M}_0 \ \mathcal{L}^1\text{-a.e. in } (0,1). \tag{7.19}$$

For each $n \in \mathbb{N}$ set $\delta_n := \sqrt[4]{\epsilon_n}$, and define the standard smooth mollifier $\rho_{\delta_n} \in C_c^\infty(\mathbb{R})$ by setting

$$\rho_{\delta_n}(t) := \frac{1}{\delta_n}\rho\Big(\frac{t}{\delta_n}\Big).$$

Observe that $\int_{\mathbb{R}} \rho_{\delta_n}(t)\,\mathrm{d}t = 1$, $\operatorname{supp}\rho_{\delta_n} \subset (-\delta_n, \delta_n)$, $\rho_{\delta_n} \geqslant 0$, and $\rho_{\delta_n}(-t) = \rho_{\delta_n}(t)$ for all $t \in \mathbb{R}$.

Extend u by zero outside $(0,1)$ and define for $t \in \mathbb{R}$,

$$v_n(t) := u(t)\chi_{J_n}(t), \text{ with } J_n := [2\delta_n, 1-2\delta_n],$$

and

$$u_n(t) := (\rho_{\delta_n} * v_n)(t) = \int_{\mathbb{R}} v_n(s)\,\rho_{\delta_n}(t-s)\,\mathrm{d}s.$$

We claim that $\{u_n\}_{n\in\mathbb{N}}$ satisfies (7.15)–(7.19). Since $\operatorname{supp}\rho_{\delta_n} \subset (-\delta_n, \delta_n)$, we have that $\operatorname{supp} u_n \subset [\delta_n, 1-\delta_n]$. By well-known results on mollification, $u_n \in W^{1,p}(\mathbb{R};\mathbb{R}^d) \cap C_c^\infty(\mathbb{R};\mathbb{R}^d)$,

$$\rho_{\delta_n} * u \to u \text{ in } L^p(\mathbb{R};\mathbb{R}^d) \text{ as } n \to +\infty,$$

and

$$\|\rho_{\delta_n} * (v_n - u)\|_{L^p(\mathbb{R};\mathbb{R}^d)} \leqslant \|v_n - u\|_{L^p(\mathbb{R};\mathbb{R}^d)},$$

while, by Lebesgue Dominated Convergence Theorem, together with the fact that $J_n \subset J_{n+1}$ for all $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}} J_n = (0,1)$,

$$v_n = u\chi_{J_n} \to u \text{ in } L^p(\mathbb{R};\mathbb{R}^d) \text{ as } n \to +\infty.$$

Thus, using in addition Minkowski's Inequality, we conclude that

$$\begin{aligned}
&\lim_{n\to+\infty} \|u_n - u\|_{L^p((0,1);\mathbb{R}^d)} \\
&\quad = \lim_{n\to+\infty} \|u_n - u\|_{L^p(\mathbb{R};\mathbb{R}^d)} \\
&\quad \leqslant \lim_{n\to+\infty} \big(\|\rho_{\delta_n} * v_n - \rho_{\delta_n} * u\|_{L^p(\mathbb{R};\mathbb{R}^d)} + \|\rho_{\delta_n} * u - u\|_{L^p(\mathbb{R};\mathbb{R}^d)}\big) \\
&\quad \leqslant \lim_{n\to+\infty} \big(\|v_n - u\|_{L^p(\mathbb{R};\mathbb{R}^d)} + \|\rho_{\delta_n} * u - u\|_{L^p(\mathbb{R};\mathbb{R}^d)}\big) \\
&\quad = 0,
\end{aligned}$$

which proves (7.15). We now verify that (7.16) and (7.17) are also satisfied. In fact, we have that

$$\begin{aligned}
\sup_{t\in(0,1)} |u_n(t)| &= \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \frac{1}{\delta_n} \rho\Big(\frac{t-s}{\delta_n}\Big) v_n(s)\,\mathrm{d}s \right| \leqslant \frac{1}{\delta_n} \sup_{t\in\mathbb{R}} \rho(t) \int_{2\delta_n}^{1-2\delta_n} |u(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{\delta_n} c_{u,\rho},
\end{aligned}$$

and, using Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
\sup_{t\in(0,1)} |\dot{u}_n(t)| &= \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \frac{1}{\delta_n^2} \dot{\rho}\Big(\frac{t-s}{\delta_n}\Big) v_n(s)\,\mathrm{d}s \right| \leqslant \frac{1}{\delta_n^2} \sup_{t\in\mathbb{R}} |\dot{\rho}(t)| \int_{2\delta_n}^{1-2\delta_n} |u(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{\delta_n^2} c_{u,\rho},
\end{aligned}$$

which, recalling that $\delta_n = \sqrt[4]{\epsilon_n}$, yields (7.16) and (7.17). Finally, we show that (7.18) and (7.19) also hold. We have that for all $t \in \mathbb{R}$,

$$\sum_{i=1}^d u_n^i(t) = \int_{\mathbb{R}} \bigg(\sum_{i=1}^d v_n^i(s) \bigg) \rho_{\delta_n}(t-s)\,\mathrm{d}s = \int_{2\delta_n}^{1-2\delta_n} \bigg(\sum_{i=1}^d u^i(s) \bigg) \rho_{\delta_n}(t-s)\,\mathrm{d}s = 0$$

which proves (7.18). On the other hand, for all $i, j \in I_d$ and for all $t \in (0,1)$, we have that

$$\begin{aligned}
\frac{u_n^i(t) - u_n^j(t)}{2} &= \int_{\mathbb{R}} \frac{v_n^i(s) - v_n^j(s)}{2} \rho_{\delta_n}(t-s)\,\mathrm{d}s \\
&= \int_{2\delta_n}^{1-2\delta_n} \frac{u^i(s) - u^j(s)}{2} \rho_{\delta_n}(t-s)\,\mathrm{d}s \\
&\leqslant \bar{M}_0 \int_{2\delta_n}^{1-2\delta_n} \rho_{\delta_n}(t-s)\,\mathrm{d}s \leqslant \bar{M}_0,
\end{aligned}$$

from which we obtain (7.19) by taking the maximum over $i \in I_d$ and then the maximum over $j \in I_d$.

Substep 2.2. We prove that the sequence constructed in Substep 2.1 is such that

$$\begin{aligned}
&\int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\,\mathrm{d}t \\
&\qquad = \lim_{n\to+\infty} \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda\mathbf{1}\big)\,\mathrm{d}t.
\end{aligned} \tag{7.20}$$

By the local Lipschitz continuity of $\tilde{h}$, together with the fact that $\|u(\cdot)\|_\sharp$, $\|u_n(\cdot)\|_\sharp \leqslant \bar{M}_0$ $\mathcal{L}^1$-a.e. in $(0,1)$, and for all $n \in \mathbb{N}$, we can find a positive constant c, only depending on $\bar{M}_0$ and d, such that for all $n \in \mathbb{N}$ one has

$$\sup_{t\in(0,1)} \big(|\tilde{\mathbf{h}}(u_n(t))| + |\tilde{\mathbf{h}}(u(t))|\big) \leqslant c.$$

Using in addition (7.17), there is a positive constant $\tilde{c}$, only depending on R_0, M_0, $\bar{M}_0$ and d, such that

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \Big(&\|\dot{w}\|_{L^\infty(0,1)} + \epsilon_n \|\dot{u}_n\|_{L^\infty((0,1);\mathbb{R}^d)} + \|\tilde{\mathbf{h}}(u_n)\|_{L^\infty((0,1);\mathbb{R}^d)} \\
&\qquad + \|\tilde{\mathbf{h}}(u)\|_{L^\infty((0,1);\mathbb{R}^d)} + |\lambda|\Big) \leqslant \tilde{c}.
\end{aligned}$$

In view of the local Lipschitz continuity of F^* we can find another constant $\bar{c}$, only depending on $\tilde{c}$, such that for $\mathcal{L}^1$-a.e. $t \in (0,1)$,

$$\begin{aligned}
&\big|F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda\mathbf{1}\big) - F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\big| \\
&\qquad \leqslant \bar{c}\big|\epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \tilde{\mathbf{h}}(u(t))\big| \leqslant \bar{c}\big(\epsilon_n |\dot{u}_n(t)| + L|u_n(t) - u(t)|\big),
\end{aligned}$$

where in the last inequality we used (7.1). By (7.17) and (7.15), we conclude that

$$\begin{aligned}
\lim_{n\to+\infty} \int_0^1 &\big|F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda\mathbf{1}\big) \\
&\qquad - F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda\mathbf{1}\big)\big|\,\mathrm{d}t = 0,
\end{aligned}$$

which yields (7.20).

Substep 2.3. We establish (7.14).

Define $w_n := w$ and $\lambda_n := \lambda$ for all $n \in \mathbb{N}$, and let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence constructed in Substep 2.1. For each $n \in \mathbb{N}$, set $\delta_n := \sqrt{\epsilon_n}$, and let $\phi_n \in C_c^\infty(\mathbb{R};[0,1])$ be a smooth cut-off function such that

$$\begin{cases} \phi_n = 1 \quad \text{in } [0, 1-2\delta_n], \\ \phi_n = 0 \quad \text{in } [1-\delta_n, +\infty), \\ \|\phi_n\|_{L^\infty(\mathbb{R})} \leqslant \dfrac{2}{\delta_n}. \end{cases}$$

Now we define

$$v_n(t) := u_n(t)\phi_n(t) + (1-\phi_n(t))\psi, \quad t \in (0,1), n \in \mathbb{N}.$$

We have that for all $n \in \mathbb{N}$,

$$v_n \in W^{1,p}((0,1);\mathbb{R}^d) \cap C^\infty([0,1];\mathbb{R}^d), \quad v_n(1) = \psi,$$

and so, $v_n \in \mathcal{A}_\psi$. Moreover,

$$v_n \to u \text{ in } L^p((0,1);\mathbb{R}^d) \text{ as } n \to +\infty, \tag{7.21}$$

due to (7.15) and to the pointwise convergence $\phi_n \to 1$ in (0,1) together with Lebesgue Dominated Convergence Theorem. Also,

$$\sum_{i=1}^d v_n^i(t) = \phi_n(t) \sum_{i=1}^d u_n^i(t) + (1-\phi_n(t)) \sum_{i=1}^d \psi^i = 0,$$

where we used the fact that $\sum_{i=1}^d \psi^i = 0$ and $\sum_{i=1}^d u_n^i(\cdot) = 0$ $\mathcal{L}^1$-a.e. in $(0,1)$. Furthermore,

$$\dot{v}_n(t) = \dot{u}_n(t)\phi_n(t) + \dot{\phi}_n(t)(u_n(t) - \psi),$$

and so, by (7.16) and (7.17),

$$\epsilon_n \|\dot{v}_n(t)\|_{L^\infty((0,1);\mathbb{R}^d)} \leqslant \sqrt{\epsilon_n} c_{u.\rho} + 2\sqrt[4]{\epsilon_n} c_{u,\rho} + 2\sqrt{\epsilon_n}|\psi|.$$

Thus, for all $n \in \mathbb{N}$ large enough we have that

$$\int_0^1 |\epsilon_n \dot{v}_n(t)|^p \, \mathrm{d}t \leqslant M_0.$$

In particular, $\epsilon_n v_n \rightharpoonup 0$ weakly in $W^{1,p}((0,1);\mathbb{R}^d)$ as $n \to +\infty$. Consequently, by the continuity of the trace operator, $\epsilon_n v_n(0) \to 0$ in $\mathbb{R}^d$ as $n \to +\infty$. We observe further that since $0 \leqslant \phi_n \leqslant 1$ and $\|\psi\|_\sharp \leqslant \bar{M}_0$, and due to (7.19), we get for all $i, j \in I_d$, $t \in (0,1)$, and $n \in \mathbb{N}$,

$$\begin{aligned} \left| \frac{v_n^i(t) - v_n^j(t)}{2} \right| &\leqslant \phi_n(t) \left| \frac{u_n^i(t) - u_n^j(t)}{2} \right| + (1-\phi_n(t)) \left| \frac{\psi^i - \psi^j}{2} \right| \\ &\leqslant \phi_n(t)\bar{M}_0 + (1-\phi_n(t))\bar{M}_0 = \bar{M}_0. \end{aligned}$$

Arguing as before, we obtain $\|v_n(\cdot)\|_\sharp \leqslant \bar{M}_0$ $\mathcal{L}^1$-a.e. in $(0,1)$. Consequently,

$$
\begin{aligned}
&\limsup_{n\to+\infty} \mathcal{F}_{\epsilon_n}(v_n, w_n, \lambda_n) \\
&\quad = \limsup_{n\to+\infty} \bigg(\int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{v}_n(t) + \tilde{\mathbf{h}}(v_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \epsilon_n \theta_0 \cdot v_n(0) - \lambda \bigg) \\
&\quad = \limsup_{n\to+\infty} \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{v}_n(t) + \tilde{\mathbf{h}}(v_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t - \lambda.
\end{aligned}
\tag{7.22}
$$

Finally, we observe that

$$
\begin{aligned}
&\int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{v}_n(t) + \tilde{\mathbf{h}}(v_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t \\
&\qquad = \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t + E_n,
\end{aligned}
\tag{7.23}
$$

where

$$
\begin{aligned}
E_n := &- \int_{1-2\delta_n}^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{u}_n(t) + \tilde{\mathbf{h}}(u_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t \\
&\quad + \int_{1-2\delta_n}^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon_n \dot{v}_n(t) + \tilde{\mathbf{h}}(v_n(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t
\end{aligned}
$$

so that, taking into account the local Lipschitz continuity of F^* and $\tilde{h}$, taking into account the bounds satisfied by $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, and arguing as in Substep 2.2, we obtain

$$
|E_n| \leqslant \int_{1-2\delta_n}^1 \bar{c}_1 \,\mathrm{d}t = 2\bar{c}_1 \delta_n = 2\bar{c}_1 \sqrt{\epsilon_n},
$$

with $\bar{c}_1 \in \mathbb{R}^+$ independent of $n \in \mathbb{N}$. Thus, $|E_n| \to 0$ as $n \to +\infty$, which together with (7.22), (7.23), (7.20) and (7.21), concludes the proof of Substep 2.3 and of Theorem 7.1. $\square$

Corollary 7.2. *For each $\epsilon > 0$, let $\mathcal{G}_\epsilon : W^{1,p}((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ and $\mathcal{G}_0 : L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ be the functionals defined by*

$$
\mathcal{G}_\epsilon(u, w, \lambda) := \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \epsilon \dot{u}(t) + \tilde{\mathbf{h}}(u(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t - \epsilon \theta_0 \cdot u(0) - \lambda,
$$

$(u, w, \lambda) \in W^{1,p}((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$, and

$$
\mathcal{G}_0(u, w, \lambda) := \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t - \lambda,
$$

$(u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$, *respectively. Consider also the sets*

$$\Phi_\epsilon := \Bigg\{ (u,w,\lambda) \in W^{1,p}((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}\colon \\ u(1) = \psi,\ \|u(\cdot)\|_\sharp \leqslant \bar{M}_0 \text{ and } \sum_{i=1}^d u^i(\cdot) = 0 \ \mathcal{L}^1\text{-a.e. in } (0,1), \\ \max\Big\{ \int_0^1 |\epsilon \dot{u}(t)|^p \,\mathrm{d}t, \|\dot{w}\|_{L^\infty(0,1)} \Big\} \leqslant M_0,\ |\lambda| \leqslant R_0 \Bigg\},$$

and

$$\Phi_0 := \Bigg\{ (u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}\colon \\ \|u(\cdot)\|_\sharp \leqslant \bar{M}_0 \text{ and } \sum_{i=1}^d u^i(\cdot) = 0 \ \mathcal{L}^1\text{-a.e. in } (0,1), \\ \|\dot{w}\|_{L^\infty(0,1)} \leqslant M_0,\ |\lambda| \leqslant R_0 \Bigg\}.$$

Then

$$\min_{(u,w,\lambda)\in\Phi_0} \mathcal{G}_0(u,w,\lambda) = \lim_{\epsilon\to 0^+} \inf_{(u,w,\lambda)\in\Phi_\epsilon} \mathcal{G}_\epsilon(u,w,\lambda).$$

Moreover, if for each $\epsilon > 0$ $(u_\epsilon, w_\epsilon, \lambda_\epsilon)$ is a minimizer of $\mathcal{G}_\epsilon$ in Φ_ϵ (or, more generally, a δ_ϵ-minimizer, where $\{\delta_\epsilon\}_{\epsilon>0}$ is a sequence of positive numbers converging to 0) and (u,w,λ) is a cluster point of $\{(u_\epsilon, w_\epsilon, \lambda_\epsilon)\}_{\epsilon>0}$, then (u,w,λ) is a minimizer of $\mathcal{G}_0$ in Φ_0, and

$$\mathcal{G}_0(u,w,\lambda) = \limsup_{\epsilon\to 0^+} \mathcal{G}_\epsilon(u_\epsilon, w_\epsilon, \lambda_\epsilon).$$

If $\{(u_\epsilon, w_\epsilon, \lambda_\epsilon)\}_{\epsilon>0}$ weakly converges to (u,w,λ) in $L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$, then (u,w,λ) is a minimizer of $\mathcal{G}_0$ in Φ_0, and

$$\mathcal{G}_0(u,w,\lambda) = \lim_{\epsilon\to 0^+} \mathcal{G}_\epsilon(u_\epsilon, w_\epsilon, \lambda_\epsilon).$$

Proof. Let $\mathcal{F}_\epsilon$, $\epsilon > 0$, and $\mathcal{F}_0$ be the functionals given by (7.2) and (7.3), respectively. If we prove that $\{\mathcal{F}_\epsilon\}_{\epsilon>0}$ is equi-coercive in the weak topology of $L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}$, then Corollary 7.2 is an immediate consequence of Theorems 6.4 and 7.1, observing that

$$\min_{(u,w,\lambda)\in L^p((0,1);\mathbb{R}^d)\times W_0^{1,p}(0,1)\times\mathbb{R}} \mathcal{F}_0(u,w,\lambda) = \min_{(u,w,\lambda)\in\Phi_0} \mathcal{G}_0(u,w,\lambda),$$
$$\inf_{(u,w,\lambda)\in L^p((0,1);\mathbb{R}^d)\times W_0^{1,p}(0,1)\times\mathbb{R}} \mathcal{F}_\epsilon(u,w,\lambda) = \inf_{(u,w,\lambda)\in\Phi_\epsilon} \mathcal{G}_\epsilon(u,w,\lambda).$$

Fix $s \in \mathbb{R}$. We claim that there exists a constant $\mathcal{C} > 0$, only depending on R_0, M_0, $\bar{M}_0$, and d, such that for all $\epsilon > 0$, one has

$$\{(u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}\colon \mathcal{F}_\epsilon(u,w,\lambda) \leqslant s\} \\ \subset \{(u,w,\lambda) \in L^p((0,1);\mathbb{R}^d) \times W_0^{1,p}(0,1) \times \mathbb{R}\colon \\ \|u\|_{L^p((0,1);\mathbb{R}^d)} + \|w\|_{W_0^{1,p}(0,1)} + |\lambda| \leqslant \mathcal{C}\}.$$

In fact, if $\mathcal{F}_\epsilon(u,w,\lambda) \leqslant s$, then, in particular, $|\lambda| \leqslant R_0$, $\|w\|^p_{W_0^{1,p}(0,1)} \leqslant 2M_0^p$, and

$$\|u(\cdot)\|_\sharp \leqslant \bar{M}_0 \ \mathcal{L}^1\text{-a.e. in } (0,1), \quad \sum_{i=1}^d u^i(t) = 0.$$

Thus, for all $j \in I_d$,

$$\begin{aligned}
\int_0^1 |du^j(t)|^p \,\mathrm{d}t &= \int_0^1 \left| \sum_{i=1}^d u^i(t) + \sum_{i=1}^d (u^j(t) - u^i(t)) \right|^p \mathrm{d}t \\
&\leqslant 2^{p-1} \left(\int_0^1 \left| \sum_{i=1}^d u^i(t) \right|^p \mathrm{d}t + \int_0^1 \left| \sum_{i=1}^d (u^j(t) - u^i(t)) \right|^p \mathrm{d}t \right) \\
&\leqslant 2^{\frac{d+1}{2}(p-1)} \bar{M}_0^p,
\end{aligned}$$

from which, together with the estimates $|\lambda| \leqslant R_0$ and $\|w\|^p_{W_0^{1,p}(0,1)} \leqslant 2M_0^p$, the claim easily follows, and, consequently, finishes the proof of Corollary 7.2. □

Remark 7.3. Let $\mathcal{G}_0$ and Φ_0 be as in Corollary 7.2. Then

$$\begin{aligned}
&\min \Big\{ \mathcal{G}_0(u,w,\lambda) \colon (u,w,\lambda) \in \Phi_0 \Big\} \\
&\quad = \min \left\{ F^*(\tilde{h}(\Delta.v,\cdot) - \lambda \mathbf{1}) - \lambda \colon v \in \mathbb{R}^d, \, \|v\|_\sharp \leqslant \bar{M}_0, \, \sum_{i=1}^d v^i = 0, \, |\lambda| \leqslant R_0 \right\},
\end{aligned} \tag{7.24}$$

which corresponds to the minimization over $|\lambda| \leqslant R_0$ of problem (3.7). As we have seen in the end of Section 3, minimizers for the latter provide stationary solutions in the sense of Definition 2.1.

To prove (7.24), we start by noticing that taking $|\lambda| \leqslant R_0$, $w \equiv 0$, and $u \in \mathbb{R}^d$ such that $\|u\|_\sharp \leqslant \bar{M}_0$ and $\sum_{i=1}^d u^i = 0$, we conclude that minimum on the left-hand side of (7.24) is less than or equal to the minimum on the right-hand side of (7.24).

Conversely, fix $(u,w,\lambda) \in \Phi_0$. Then $\int_0^1 \dot{w}(t)\,\mathrm{d}t = w(1) - w(0) = 0$. Moreover, setting $v := \int_0^1 u(t)\,\mathrm{d}t \in \mathbb{R}^d$, then $\sum_{i=1}^d v^i = 0$ and, arguing as in Theorem 7.1, $\|v\|_\sharp \leqslant \bar{M}_0$. Hence, using Jensen's inequality twice, recalling that F^* is convex and non-increasing while $\mathbf{h}$ is componentwise concave, we deduce that

$$\begin{aligned}
\mathcal{G}_0(u,w,\lambda) &= \int_0^1 F^*\big(\dot{w}(t)\mathbf{1} + \tilde{\mathbf{h}}(u(t)) - \lambda \mathbf{1}\big)\,\mathrm{d}t - \lambda \\
&\geqslant F^*\left(\int_0^1 \tilde{\mathbf{h}}(u(t))\,\mathrm{d}t - \lambda \mathbf{1} \right) - \lambda \\
&\geqslant F^*\big(\tilde{\mathbf{h}}(v) - \lambda \mathbf{1}\big) - \lambda \\
&\geqslant \min \left\{ F^*(\tilde{h}(\Delta.v,\cdot) - \lambda \mathbf{1}) - \lambda \colon v \in \mathbb{R}^d, \, \|v\|_\sharp \leqslant \bar{M}_0, \, \sum_{i=1}^d v^i = 0, \, |\lambda| \leqslant R_0 \right\},
\end{aligned}$$

from which the conclusion follows by taking the infimum over $(u,w,\lambda) \in \Phi_0$.

Assume now that F^* is strictly convex and that $\tilde{h}$ is strictly concave in $\mathbb{R}^d \backslash \mathbb{R}$, that is, for all $0 < \mu < 1$ we have

$$\tilde{h}(\mu \Delta_i u + (1-\mu)\Delta_i v, i) = \mu \tilde{h}(\Delta_i u, i) + (1-\mu)\tilde{h}(\Delta_i v, i)$$

implies $u = v + k\mathbf{1}$, for some $k \in \mathbb{R}$. Using once again Jensen's inequality, we conclude that a solution $(u, w, \lambda) \in \Phi_0$ to

$$\min\left\{\mathcal{G}_0(u, w, \lambda)\colon\ (u, w, \lambda) \in \Phi_0\right\}$$

is such that (w, u) does not depend on time. Thus, in this setting, Corollary 7.2 establishes in addition convergence of solutions of (5.1)–(5.2) to stationary solutions in the sense of Definition 2.1.

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