CHARACTERIZATION OF POLYNOMIALS AND HIGHER-ORDER SOBOLEV SPACES IN TERMS OF NONLOCAL FUNCTIONALS INVOLVING DIFFERENCE QUOTIENTS

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Abstract. The aim of this paper, which deals with a class of singular functionals involving difference quotients, is twofold: deriving suitable integral conditions under which a measurable function is polynomial and stating necessary and sufficient criteria for an integrable function to belong to a kth-order Sobolev space. One of the main theorems is a new characterization of $W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and $p \in (1, +\infty)$, for arbitrary open sets $\Omega \subset \mathbb{R}^n$. In particular, we provide natural generalizations of the results regarding Sobolev spaces summarized in Brézis' overview article [*Russ. Math. Surv.* **57** (2002), pp. 693-708] to the higher-order case, and extend the work by Borghol [*Asymptotic Anal.* **51** (2007), pp. 303-318] to a more general setting.

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1. Introduction

The main motivation for this work was the overview paper by Brézis [5] on the question of how to recognize constant functions through integral conditions, where the author essentially collects the basic ideas of the work by Bourgain, Brézis & Mironescu [3, 4] regarding the characterization of Sobolev and BV spaces in terms of singular integrals. Interestingly, this problem turns out to be relevant for improved models in image and signal processing due to its close relation with the nonlocal functionals recently proposed by Gilboa & Osher [8, 9] to avoid the undesirable staircase effect (see [11, 1]). Further applications and connections pointed out in [5] include lifting maps with values on the unit sphere, degree theory for classes of discontinuous functions, and the space VMO of functions with vanishing mean oscillation.

In what follows, we focus on extending the results of [5] regarding the characterization of first-order Sobolev spaces to the higher-order case. This will also yield criteria for recognizing polynomials.

Let us start by recalling one of the results in [5] concerning integral conditions only satisfied by constant functions.

Proposition 1.1 ([5, Proposition 2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $f : \Omega \to \mathbb{R}$ be a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+p}} \, \mathrm{d}y \, \mathrm{d}x < +\infty \tag{1.1}$$

for some $p \in [1, +\infty)$. Then f is a constant function.

In [5], the proof of Propositions 1.1 follows from a more general result (see [5, Theorem 1]), which asserts that if $(\rho_\varepsilon)_\varepsilon$ is a family of mollifiers such that for all $\varepsilon > 0$,

$$\rho_\varepsilon \in L^1_{\mathrm{loc}}(0, +\infty), \quad \rho_\varepsilon \geq 0, \quad \int_{\mathbb{R}^n} \rho_\varepsilon(|h|) \, \mathrm{d}h = 1, \tag{1.2}$$

and for all $\gamma > 0$,

$$\lim_{\varepsilon \to 0^+} \int_{\{|h| > \gamma\}} \rho_\varepsilon(|h|) \, \mathrm{d}h = 0, \tag{1.3}$$

then the only measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\lim_{\varepsilon\to 0^+} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)\, \mathrm{d}y\, \mathrm{d}x = 0, \tag{1.4}$$

with $p \in [1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ open and connected, are the constant functions. Provided (1.1) holds true, then (1.4) is satisfied for the specific choice of mollifiers

$$\rho_\varepsilon(r) := \frac{\varepsilon}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\, r^{n-\varepsilon}} \chi_{(0,1)}(r), \qquad r \in (0, +\infty), \quad \varepsilon > 0.$$

Moreover, if Ω is open and connected and f is constant on any ball contained in Ω, then f is constant in Ω; thus, it suffices to assume that (1.1) or (1.4) hold on any such ball.

In turn, the proof of the sufficiency of condition (1.4) for f being constant is based on a new characterization of Sobolev and BV spaces for *smooth, bounded* domains summarized in the following result, which was first proved by Bourgain, Brézis & Mironescu [3, 4] in the Sobolev setting and in the BV setting for $n = 1$, and by Dávila [6] in the BV setting for any $n \in \mathbb{N}$.

Theorem 1.2 ([5, Theorems 2 and 3, Remark 7]**; see also** [3, 4, 6])**.** *Let $\Omega \subset \mathbb{R}^n$ be either a smooth, bounded, and open set or the whole space $\mathbb{R}^n$. Let $f \in L^p(\Omega)$, with $p \in [1, +\infty)$, and let $(\rho_\varepsilon)_\varepsilon$ be a family of mollifiers satisfying* (1.2) *and* (1.3)*. Then*

$$\begin{cases} f \in W^{1,p}(\Omega) & \text{if } p > 1, \\ f \in BV(\Omega) & \text{if } p = 1, \end{cases}$$

if and only if

$$\limsup_{\varepsilon\to 0^+} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)\, \mathrm{d}y\, \mathrm{d}x < +\infty, \tag{1.5}$$

in which case

$$\lim_{\varepsilon\to 0^+} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)\, \mathrm{d}y\, \mathrm{d}x = K_{n,p} \int_\Omega |\nabla f(x)|^p\, \mathrm{d}x, \tag{1.6}$$

where $K_{n,p}$ is a constant only depending on n and p, and for $p = 1$, $\int_\Omega |\nabla f(x)|\, \mathrm{d}x$ denotes the total variation of the distributional derivative of f.

Several questions related to Proposition 1.1 and Theorem 1.2 were raised in [5]. Let us emphasize here three of them (see [5, Remark 1, Problem 2, Remark 5]).

Question A. Is there a direct, elementary proof of Proposition 1.1, that is, without involving Sobolev or BV spaces?

Question B. Given a smooth, connected, and open set $\Omega \subset \mathbb{R}^n$ and a continuous function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$ and $\omega(t) > 0$ for all $t > 0$, what additional conditions (if any) on ω and/or on f should be imposed so that replacing (1.1) with

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{1}{|x-y|^n}\, \mathrm{d}y\, \mathrm{d}x < +\infty, \tag{1.7}$$

still entails that f is constant?

Question C. If $\Omega \subset \mathbb{R}^n$ is an open and bounded set whose boundary $\partial\Omega$ is not smooth, it is possible to construct an example of a function $f \in W^{1,p}(\Omega)$ for which (1.5) fails. For such sets Ω, what condition should replace (1.5) in order to derive an analogous result to Theorem 1.2?

Question A was addressed for $p = 1$ by De Marco, Mariconda & Solimini [7], who provide two different arguments with no connections to BV or Sobolev spaces. Their first proof uses a convolution argument, while the second one is based on a kind of non-smooth mean value theorem.

Question B was discussed in detail by Ignat [10]. For instance, in [10, Theorem 1.3] it is proved that if ω is, in addition, such that $\liminf_{t\to+\infty} \omega(t)/t > 0$, then the only measurable functions satisfying (1.7) are constants. We refer to [10] for other statements related to Question B.

Regarding Question C, an answer was given by Leoni & Spector [11], who provide a characterization of the spaces $W^{1,p}(\Omega)$, $p \in (1,+\infty)$, and $BV(\Omega)$ for *arbitrary* open sets $\Omega \subset \mathbb{R}^n$ (hence not necessarily smooth or bounded) by replacing condition (1.5) with

$$\lim_{\lambda\to 0^+} \limsup_{\varepsilon\to 0^+} \int_{\Omega_\lambda}\int_{\Omega_\lambda} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)\,\mathrm{d}y\,\mathrm{d}x < +\infty, \tag{1.8}$$

where $\Omega_\lambda := \{x \in \Omega\colon |x| < 1/\lambda,\ \mathrm{dist}(x, \partial\Omega) > \lambda\}$ (see [11, Theorems 1.5 and 1.9] as well as [12] for the full result).

The main goal of this paper is to extend Proposition 1.1, or more generally [5, Theorem 1] (cf. (1.4)), and Theorem 1.2 to the higher-order case addressing simultaneously Questions A, B, and C. Before stating our main results, we start by mentioning that a generalization of Theorem 1.2 to the context of Sobolev and BV spaces of higher-order, which yields criteria for recognizing polynomials, was studied by Borghol [2]. However, the results in [2] only hold for bounded, open, smooth, and convex sets $\Omega \subset \mathbb{R}^n$. Here, besides treating the case of arbitrary open sets, we present different integral conditions involving higher-order difference quotients that require only weak assumptions on the function to be characterized, and that seem better suited for the application in image denoising models with nonlocal regularization terms in the spirit of [8, 9, 1].

Given a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, we define the forward differences $\Delta_h f : \mathbb{R}^n \to \mathbb{R}$ by $\Delta_h f(x) := f(x + h) - f(x)$ for x, $h \in \mathbb{R}^n$. Setting $\Delta_h^1 f := \Delta_h f$, kth-order forward differences are defined inductively by

$$\Delta_h^k f(x) := \Delta_h\big(\Delta_h^{k-1} f(x)\big), \quad x,\, h \in \mathbb{R}^n,\ \ k \in \mathbb{N},\ \ k \geq 2. \tag{1.9}$$

The following proposition, which is a consequence of Theorem 1.4 below, is a natural generalization of Proposition 1.1 to the higher-order case for functions defined on $\mathbb{R}^n$.

Proposition 1.3. *Let $k \in \mathbb{N}$, $p \in [1,+\infty)$, and $f : \mathbb{R}^n \to \mathbb{R}$ a measurable function such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Delta_h^k f(x)|^p}{|h|^{n+kp}}\,\mathrm{d}h\,\mathrm{d}x < +\infty. \tag{1.10}$$

Then f coincides with a polynomial of degree at most $k - 1$ almost everywhere in $\mathbb{R}^n$.

While in the case $k = 1$ measurability of a function f satisfying (1.10) implies its local integrability almost immediately (see Remark 3.2 (ii)), for $k > 1$ this issue is non-trivial and to our knowledge has not been treated before in the context of higher orders. In Section 3 we give detailed proofs, showing in particular the local integrability of a measurable function satisfying (1.10). The arguments use ideas from Stein & Zygmund [15].

We now state a refined and more flexible version of Proposition 1.3 inspired by (1.4) and Question B. Throughout the paper, we assume that

$$\omega : [0,+\infty) \to [0,+\infty) \text{ is a strictly increasing, convex function with } \omega(0) = 0. \tag{1.11}$$

Theorem 1.4. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ a connected open set, ω a function as in* (1.11)*, and $(\rho_\varepsilon)_\varepsilon$ a family of mollifiers satisfying* (1.2) *and* (1.3)*. Suppose that $f : \Omega \to \mathbb{R}$ is a locally integrable function such that for every $x_0 \in \Omega$, there exists $r_0 > 0$ with $B(x_0, (k+1)r_0) \subset \Omega$ and*

$$\lim_{\varepsilon\to 0^+} \int_{B(x_0,r_0)}\int_{B(0,r_0)} \omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x = 0. \tag{1.12}$$

Then f coincides almost everywhere in Ω with a polynomial of degree smaller than or equal to $k - 1$.

Remark 1.5. (i) *Local character of* (1.12)*.* We note that if $x_0 \in \Omega$ and $r_0 > 0$ are as in Theorem 1.4, then the integrand of the double integral on the left-hand side of (1.12) is well-defined for every $(x, h) \in B(x_0, r_0) \times B(0, r_0)$. As it will become clear within the proof of Theorem 1.4, condition (1.12) implies that f coincides with a polynomial of degree at most $k - 1$ almost everywhere in a neighborhood of x_0. As Ω is open and connected, a covering argument yields the same conclusion in Ω.

(ii) *Local integrability vs. measurability of f.* Under an extra assumption on $(\rho_\varepsilon)_\varepsilon$, which we call hypothesis (H) and which is stated in Definition 3.3, the condition $f \in L^1_{\rm loc}(\Omega)$ in Theorem 1.4 can be weakened by requiring only measurability of f, see Corollary 3.5.

(iii) *Comparison with* [10, Theorem 1.3]. Concerning the assumptions on the function ω, our arguments rely crucially on the monotonicity and convexity of ω. Since these hypotheses imply that $\liminf_{t\to+\infty}\omega(t)/t > 0$, Theorem 1.4 can be viewed as a particular case of [10, Theorem 1.3] for $k = 1$ (cf. (1.7)).

One way to prove Theorem 1.4 - at least in the case where ω features standard p-growth with $p \in [1, +\infty)$ - is by an excursion through the theory of Sobolev spaces, that is, by considering it a corollary of Theorem 1.6 below (see also Remark 5.2 for the case $p = 1$). Alternatively, we provide an independent proof using only elementary arguments in Section 4. We would like to stress that these arguments differ from the ones in [7], for which reason our proof can be seen as another answer to Question A valid for any $k \in \mathbb{N}$.

Finally, the next theorem provides a characterization of higher-order Sobolev spaces of functions defined on *arbitrary* open subsets of $\mathbb{R}^n$. It extends Theorem 1.2 and addresses Question C. For an explanation of the special notation used here we refer to Section 2.

Theorem 1.6. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ an open set, $(\rho_\varepsilon)_\varepsilon$ a family of mollifiers satisfying* (1.2) *and* (1.3)*, and $f \in L^p_{\rm loc}(\Omega)$ for some $p \in (1, +\infty)$. Assume that, in addition to* (1.11)*, ω has p-growth, i.e.,*

$$mt^p \le \omega(t) \le Mt^p \tag{1.13}$$

for all $t \in [0, +\infty)$, with $0 < m \le M$. For any $x \in \Omega$, let $r_x \in (0, +\infty]$ denote $r_x := \operatorname{dist}(x, \partial\Omega)/k$. Then $f \in W^{k,p}_{\rm loc}(\Omega)$ with $D^k f \in L^p(\Omega; \mathbb{R}^{n^k})$ if and only if

$$\limsup_{\varepsilon\to0^+} \int_\Omega \int_{B(0,r_x)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x < +\infty, \tag{1.14}$$

in which case the following equality and bounds hold [1] *:*

$$\begin{aligned}\lim_{\varepsilon\to0^+} &\int_\Omega \int_{B(0,r_x)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &= \int_\Omega \fint_{\mathcal{S}^{n-1}} \omega\big(|\sigma_k(h)\nabla(D^{k-1}f)(x)\,h|\big)\, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}x,\end{aligned} \tag{1.15}$$

and

$$\begin{aligned} m\,\overline{K}_{n,p,k} \int_\Omega |D^k f(x)|^p\, \mathrm{d}x &\le \int_\Omega \fint_{\mathcal{S}^{n-1}} \omega\big(|\sigma_k(h)\nabla(D^{k-1}f)(x)\,h|\big)\, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}x \\ &\le M \int_\Omega |D^k f(x)|^p\, \mathrm{d}x,\end{aligned} \tag{1.16}$$

where $0 < \overline{K}_{n,p,k} \le 1$ is a constant depending on n, p, and k.

Remark 1.7. (i) *Basic choice for ω.* The most common choice for ω is to set $\omega(t) := t^p$ for $t \in [0, +\infty)$. In this case (1.15) specializes to

$$\lim_{\varepsilon\to0^+} \int_\Omega \int_{B(0,r_x)} \frac{|\Delta^k_h f(x)|^p}{|h|^{kp}} \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x = \int_\Omega \fint_{\mathcal{S}^{n-1}} |\sigma_k(h)\nabla(D^{k-1}f)(x)\,h|^p\, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}x,$$

and (1.16) holds with $m = M = 1$.

(ii) *More general growth conditions for ω.* If Ω has finite measure, condition (1.13) may be replaced by

$$mt^p - c \le \omega(t) \le Mt^p + C$$

for all $t \in [0, +\infty)$, with $0 < m \le M$ and $c, C \ge 0$. In this case, (1.15) remains unchanged, while a lower bound and an upper bound on (1.16) are given, respectively, by $m\overline{K}_{n,p,k}\int_\Omega |D^k f(x)|^p\, \mathrm{d}x - c|\Omega|$ and $M\int_\Omega |D^k f(x)|^p\, \mathrm{d}x + C|\Omega|$.

[1] As we will detail in Section 2, $\sigma_k(h)\nabla(D^{k-1}f)(x)\,h = \sum_{i_1,\dots,i_k=1}^n \frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(x)\,h_{i_1}\cdots h_{i_k}$.

(iii) *Local integrability vs. measurability of f.* In view of Corollary 3.5, we can take a measurable function f instead of $f \in L^p_{\mathrm{loc}}(\Omega)$ as a starting point in Theorem 1.6, provided the sequence of mollifiers satisfies (H) (cf. Definition 3.3).

(iv) *First-order Sobolev spaces.* If $k = 1$, then the constant $\overline{K}_{n,p,k}$, which follows from a minimization problem (see (5.1)), can be computed explicitly and we find that it equals the constant $K_{n,p}$ in Theorem 1.2 (see Remark 5.1). Moreover, having in mind that $\sigma_1(h) = 1$ and $D^0 f = f$, for the choice $\omega(t) = t^p$, $t \in [0, +\infty)$, the inequalities in (1.16) may be replaced by the identity

$$\int_\Omega \fint_{\mathcal{S}^{n-1}} |\nabla f(x)\, h|^p \, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}x = K_{n,p} \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x.$$

Hence, our result provides another integral condition, namely (1.14), which allows recovering the semi-norm $K_{n,p} \int_\Omega |\nabla f(x)|^p \, \mathrm{d}x$ on $W^{1,p}(\Omega)$ for arbitrary open sets $\Omega \subset \mathbb{R}^n$ and is different from (1.5) and (1.8) if $\Omega \neq \mathbb{R}^n$. For $\Omega = \mathbb{R}^n$ and $k = 1$, a change of variables transforms the left-hand side of (1.14) into

$$\begin{aligned} &\limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega\left(\frac{|f(x+h) - f(x)|}{|h|}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &\qquad = \limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\, \mathrm{d}y\, \mathrm{d}x. \end{aligned}$$

We also refer to the work by Nguyen [13, 14] for another characterization of $W^{1,p}(\mathbb{R}^n)$.

(v) *Equivalent norms for higher-order Sobolev spaces.* Note that (1.15) provides an equivalent semi-norm on $W^{k,p}(\Omega)$ and, in view of the Gagliardo-Nirenberg interpolation inequalities, equivalent norms on $W^{k,p}(\Omega')$ for $\Omega' \subset\subset \Omega$ sufficiently regular.

(vi) *The case $p = 1$.* In this work we do not deal with the BV setting. Nevertheless, the arguments used here yield partial results in this direction, which are summarized in Remark 5.2.

While the related characterization of Sobolev spaces in [2, Theorem 4 and 5] is restricted to open, bounded, convex sets $\Omega \subset \mathbb{R}^n$ with smooth boundary only, Theorem 1.6 gives necessary and sufficient conditions for Sobolev functions on *general* open sets. This particularly includes the case $\Omega = \mathbb{R}^n$.

In [2] the author uses as integrands a special type of kth-order differences that are compatible with the two identical integration domains Ω. Then, for the sake of well-definedness Ω needs to be convex. Precisely, the counterpart of (1.14) in [2] reads

$$\limsup_{\varepsilon \to 0^+} \int_\Omega \int_\Omega \left| \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{(k-j)x + jy}{k}\right)\right|^p |x-y|^{-kp} \rho_\varepsilon(|x-y|)\, \mathrm{d}y\, \mathrm{d}x < +\infty.$$

The assumption of convexity for the set Ω, though, is rather restrictive. We instead chose to work with classical forward differences - central differences would be equally suited - for the integrand. In our approach there is full generality for the first domain Ω, while the second domain of integration needs to be adapted suitably, so that symmetry with respect to the two variables is lost. Depending on the focus and the applications in mind, different kinds of double integral conditions appear reasonable. One requirement and constraint, however, when working with functions defined on a bounded set Ω is to adjust the integrand and the integration domains in such a way that the expressions are well-defined. To us there seems to be no way around a compromise between structure or symmetry and generality.

This paper is organized as follows. After introducing some notation related to difference quotients and Taylor series in Section 2, we give a useful integral condition under which a measurable function is locally integrable in Section 3 that allows removing the local integrability hypotheses in Theorems 1.4 and 1.6 provided the mollifiers satisfy an additional hypothesis. Section 4 is devoted to the proof of Theorem 1.4 and an immediate consequence formulated in Proposition 1.3. In Section 4, Theorem 1.4 will be proved only using elementary arguments. Finally, in the last section we prove Theorem 1.6 and, as a corollary, we provide an alternative proof of Theorem 1.4.

2. Notation and preliminaries

Throughout these notes let $\mathbb{M}_{mn}$ be the set of $m \times n$ matrices with entries in $\mathbb{R}$. For $F \in \mathbb{M}_{mn}$ we denote by $|F|$ the Frobenius norm of F. The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is written as $|E|$, whereas $\mathcal{H}^{n-1}(E)$ stands for its $(n-1)$-dimensional Hausdorff measure. Moreover, χ_E is the characteristic function of a set $E \subset \mathbb{R}^n$, meaning that $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise. For $x \in \mathbb{R}^n$ and $r > 0$ let $B(x, r)$ stand for the open ball in $\mathbb{R}^n$ of radius r around x. The unit sphere in $\mathbb{R}^n$ is referred to as $\mathcal{S}^{n-1}$.

For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the kth-order forward differences introduced in (1.9) are alternatively given by the explicit formula

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh), \qquad x,\, h \in \mathbb{R}^n, \tag{2.1}$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$ for $j = 0, \dots, k$. If $k = 2$, for instance, we have $\Delta_h^2 f(x) = f(x + 2h) - 2f(x + h) + f(x)$ for $x, h \in \mathbb{R}^n$.

If f is smooth, i.e., $f \in C^{k+1}(\mathbb{R}^n)$, the Taylor expansion of f about a fixed $x \in \mathbb{R}^n$ is given by

$$f(x + h) = f(x) + f'(x)h + \frac{1}{2!} f''(x)h^2 + \cdots + \frac{1}{k!} f^{(k)}(x)h^k + R^{(k)}(h; x), \tag{2.2}$$

where for $h \in \mathbb{R}^n$,

$$f^{(k)}(x)h^k := \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(x)\, h_{i_1} \cdots h_{i_k}.$$

Here $R^{(k)}$ denotes the Taylor remainder of f, which can be represented as

$$R^{(k)}(h; x) = \frac{1}{(k+1)!} f^{(k+1)}(x + \theta h)h^{k+1} \tag{2.3}$$

for some $\theta \in (0, 1)$ depending on x and h. From this it follows that

$$\Delta_h^k f(x) = f^{(k)}(x)h^k + \tilde{R}^{(k)}(h; x), \qquad h \in \mathbb{R}^n, \tag{2.4}$$

where $\tilde{R}^{(k)}(h; x)$ is a linear combination (with coefficients depending only on k) of $R^{(k)}(lh; x)$ with $l \in \{1, \dots, k\}$ and therefore,

$$\lim_{|h| \to 0^+} \frac{\tilde{R}^{(k)}(h; x)}{|h|^k} = 0. \tag{2.5}$$

For instance, if $k = 2$, since $\Delta_h^2 f(x) = f(x + 2h) - f(x) - 2(f(x + h) - f(x))$, using (2.2) yields $\Delta_h^2 f(x) = f''(x)h^2 + R^{(2)}(2h; x) - 2R^{(2)}(h; x)$ and thus

$$\tilde{R}^{(2)}(h; x) = R^{(2)}(2h; x) - 2R^{(2)}(h; x).$$

Let us rewrite (2.2) in a vectorial form that will be useful in the sequel to clarify the presentation when proving Theorem 1.6. Given a C^1-function $g : \mathbb{R}^n \to \mathbb{R}^m$, we denote by ∇g the $m \times n$ matrix

$$\nabla g = \Big(\frac{\partial g_i}{\partial x_j} \Big)_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathbb{M}_{mn}.$$

Let $\psi_{n,m} : \mathbb{M}_{mn} \to \mathbb{R}^{nm}$ be defined by

$$\psi_{n,m}(A) := (a_{11} \cdots a_{1n}\, a_{21} \cdots a_{2n} \cdots a_{m1} \cdots a_{mn})^T, \qquad A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathbb{M}_{mn}.$$

We now define recursively the functions $D^l f : \mathbb{R}^n \to \mathbb{R}^{n^l}$ and $\sigma_l : \mathbb{R}^n \to \mathbb{M}_{1n^{l-1}}$ with $l \in \{1, \cdots, k\}$ by setting

$$\begin{aligned} &D^0 f := f, \\ &D^1 f := \psi_{n,1}(\nabla f), \\ &D^l f := \psi_{n,n^{l-1}}(\nabla (D^{l-1} f)), \qquad 2 \le l \le k, \end{aligned}$$

and for $h \in \mathbb{R}^n$,

$$\sigma_1(h) := 1 \in \mathbb{R},$$
$$\sigma_l(h) := \big(h_1(\sigma_{l-1}(h)) \cdots h_n(\sigma_{l-1}(h))\big), \qquad 2 \le l \le k.$$

By construction σ_l is $(l-1)$-homogeneous and

$$|\sigma_l(h)| = |h|^{l-1}, \qquad h \in \mathbb{R}^n. \tag{2.6}$$

Under these notations, (2.2) and (2.4) can be rewritten as follows:

$$\begin{aligned} f(x+h) = f(x) + \sigma_1(h)\nabla f(x)\, h + \frac{1}{2!}\sigma_2(h)\nabla(D^1 f)(x)\, h + \cdots \\ + \frac{1}{k!}\sigma_k(h)\nabla(D^{k-1} f)(x)\, h + R^{(k)}(h;x), \end{aligned} \tag{2.7}$$

and

$$\Delta_h^k f(x) = \sigma_k(h)\nabla(D^{k-1} f)(x)\, h + \tilde{R}^{(k)}(h;x). \tag{2.8}$$

If the real-valued function f is not defined on the entire space, but only in an open set $\Omega \subset \mathbb{R}^n$, then the definitions above apply locally.

3. Local integrability

The following lemma gives an integral condition involving higher-order difference quotients that is sufficient for a measurable function to be locally integrable. This result is the essential tool that allows us to formulate Theorem 1.4 and Theorem 1.6 for measurable instead of locally integrable functions, provided the mollifiers ρ_ε satisfy the condition (H) in Definition 3.3. Our result relies on the arguments used by Stein & Zygmund [15, Lemma 13], where first-order differences were considered.

Lemma 3.1. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ an open set, ω a function as in* (1.11)*, and $f : \Omega \to \mathbb{R}$ a measurable function. Assume that $x_0 \in \Omega$ is such that there exists $r_0 > 0$ for which $B(x_0, (k+1)r_0) \subset \Omega$ and*

$$\int_{B(0,r_0)} \omega\big(|\Delta_h^k f(x)|\big)\, \mathrm{d}h < +\infty \tag{3.1}$$

for almost every $x \in B(x_0, r_0)$. Then f is locally integrable in $B(x_0, r_0)$. In particular, if the requirements above hold true for almost every $x_0 \in \Omega$, then $f \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let $x_0 \in \Omega$ and $r_0 > 0$ be as in the assumption. For $N \in \mathbb{N}$ we define

$$E_N := \Big\{ x \in B(x_0, r_0) : |f(x)| \le N, \int_{B(0,r_0)} \omega\big(|\Delta_h^k f(x)|\big)\, \mathrm{d}h \le N \Big\}.$$

Observe that $E_N \subset E_{N+1}$ for all $N \in \mathbb{N}$ and $|B(x_0, r_0) \setminus \bigcup_{N=1}^{+\infty} E_N| = 0$. By F_N we denote the set of density points of E_N, i.e.,

$$F_N := \Big\{ x \in E_N : \lim_{\eta \to 0^+} \fint_{B(x,\eta)} \chi_{E_N}(w)\, \mathrm{d}w = 1 \Big\}.$$

Owing to Lebesgue's density theorem, $|E_N \setminus F_N| = 0$ for all $N \in \mathbb{N}$. We will prove that f is integrable in a neighborhood of every point in $\bigcup_{N=1}^{+\infty} F_N$, from which we may immediately conclude the statement.

In the following let $N \in \mathbb{N}$ be fixed, and write $\mathcal{E} = E_N$ and $\mathcal{F} = F_N$. Then,

$$\int_{\mathcal{E}} \int_{B(0,r_0)} \omega(|\Delta_h^k f(x)|)\, \mathrm{d}h\, \mathrm{d}x \le N|\mathcal{E}|.$$

Performing the change of variables $x = \frac{1}{2}(y+z)$ and $h = \frac{1}{2}(y-z)$, i.e., considering on $\mathcal{M} := \big\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{1}{2}(y+z) \in \mathcal{E}, \frac{1}{2}(y-z) \in B(0,r_0)\big\} \subset \Omega \times \Omega$ the diffeomorphism

$$\varphi : \mathcal{M} \to \mathcal{E} \times B(0, r_0), \qquad (y,z) \mapsto (x,h) = \big(\tfrac{1}{2}(y+z), \tfrac{1}{2}(y-z)\big),$$

entails

$$\int_{\mathcal{M}} \omega\Big(\big|\Delta^k_{\frac{y-z}{2}} f\big(\tfrac{y+z}{2}\big)\big|\Big)\, \mathrm{d}(y,z) \le 2^n N |\mathcal{E}| < +\infty, \tag{3.2}$$

where we used the equality $|\det\nabla\varphi| = 2^{-n}$. Notice that (2.1) implies

$$
\begin{aligned}
\Delta^k_{\frac{y-z}{2}} f\big(\tfrac{y+z}{2}\big) &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} f\Big(\tfrac{(j+1)y-(j-1)z}{2}\Big) \\
&= \sum_{j=0, j\neq 1}^{k} (-1)^{k-j}\binom{k}{j} f\Big(\tfrac{(j+1)y-(j-1)z}{2}\Big) + (-1)^{k-1}k f(y)
\end{aligned}
\tag{3.3}
$$

for all $(y,z)\in\mathcal{M}$.

Let us define $\mathcal{N} := \big\{(y,z)\in\mathcal{M} : \frac{1}{2}\big((j+1)y-(j-1)z\big)\in\mathcal{E}, j=2,\dots,k\big\}$. Note that if $(y,z)\in\mathcal{N}$, then $y\in\Omega$ and $\frac{1}{2}\big((j+1)y-(j-1)z\big)\in\mathcal{E}$ for all $j\in\{0,\dots,k\}\setminus\{1\}$. Hence, in view of the properties of $\mathcal{E}$ and the fact that

$$
\omega\left(\frac{|\xi_1|}{2}\right) \leq \frac{1}{2}\omega(|\xi_2-\xi_1|) + \frac{1}{2}\omega(|\xi_2|) \tag{3.4}
$$

for all ξ_1, $\xi_2\in\mathbb{R}^n$, owing to the monotonicity and convexity of ω, we derive from (3.2) and (3.3) that

$$
\begin{aligned}
\int_{\mathcal{N}} \omega\Big(\tfrac{k}{2}|f(y)|\Big)\,\mathrm{d}(y,z) &\leq \frac{1}{2}\int_{\mathcal{N}} \omega\left(\Big|\Delta^k_{\frac{y-z}{2}} f\big(\tfrac{y+z}{2}\big) - (-1)^{k-1}k f(y)\Big|\right)\,\mathrm{d}(y,z) + 2^{n-1}N|\mathcal{E}| \\
&\leq \frac{1}{2}|\mathcal{N}|\,\omega\left(N\sum_{j=0,j\neq 1}^{k}\binom{k}{j}\right) + 2^{n-1}N|\mathcal{E}| < +\infty.
\end{aligned}
\tag{3.5}
$$

For $y\in\mathbb{R}^n$ let us set

$$
\zeta(y) := \int_{\mathbb{R}^n} \chi_{B(0,r_0)}\big(\tfrac{y-z}{2}\big)\chi_{\mathcal{E}}\big(\tfrac{y+z}{2}\big)\prod_{j=2}^{k}\chi_{\mathcal{E}}\Big(\tfrac{(j+1)y-(j-1)z}{2}\Big)\,\mathrm{d}z.
$$

It follows from (3.5) together with Fubini's theorem that

$$
\int_{\Omega}\omega\Big(\tfrac{k}{2}|f(y)|\Big)\zeta(y)\,\mathrm{d}y = \int_{\mathcal{N}}\omega\Big(\tfrac{k}{2}|f(y)|\Big)\,\mathrm{d}(y,z) < +\infty. \tag{3.6}
$$

The next step consists in deriving a lower bound on ζ. In the following we denote $\psi_{\mathcal{E}} := 1-\chi_{\mathcal{E}}$. For $y\in\mathbb{R}^n$ and $\eta>0$ one obtains

$$
\begin{aligned}
\zeta(y) &\geq \int_{B(y,\eta)} \chi_{\mathcal{E}}\big(\tfrac{y+z}{2}\big)\prod_{j=2}^{k}\chi_{\mathcal{E}}\Big(\tfrac{(j+1)y-(j-1)z}{2}\Big)\,\mathrm{d}z \\
&\geq |B(y,\eta)| - \int_{B(y,\eta)}\psi_{\mathcal{E}}\big(\tfrac{y+z}{2}\big)\,\mathrm{d}z - \sum_{j=2}^{k}\int_{B(y,\eta)}\psi_{\mathcal{E}}\Big(\tfrac{(j+1)y-(j-1)z}{2}\Big)\,\mathrm{d}z \\
&= \eta^n|B(0,1)| - 2^n\int_{B(y,\frac{\eta}{2})}\psi_{\mathcal{E}}(w)\,\mathrm{d}w - 2^n\sum_{j=2}^{k}(j-1)^{-n}\int_{B\left(y,\frac{(j-1)\eta}{2}\right)}\psi_{\mathcal{E}}(w)\,\mathrm{d}w.
\end{aligned}
\tag{3.7}
$$

The estimates in (3.7) may be proved arguing inductively, observing that if B, A_1, and A_2 are three measurable sets in $\mathbb{R}^n$, then

$$
\begin{aligned}
\int_B \chi_{A_1}(w)\chi_{A_2}(w)\,\mathrm{d}w &= \int_B \chi_{A_1}(w)\,\mathrm{d}w - \int_B \chi_{A_1}(w)\,\psi_{A_2}(w)\,\mathrm{d}w \\
&\geq \int_B \chi_{A_1}(w)\,\mathrm{d}w - \int_B \psi_{A_2}(w)\,\mathrm{d}w \\
&= |B| - \int_B \psi_{A_1}(w)\,\mathrm{d}w - \int_B \psi_{A_2}(w)\,\mathrm{d}w.
\end{aligned}
$$

Let $\bar{y}\in\mathcal{F}$, where we recall $\mathcal{F}$ is the set of density points of $\mathcal{E}$. Then,

$$
\lim_{\delta\to 0^+} \fint_{B(\bar{y},\delta)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w = 0.
$$

Consequently, for $y \in B(\bar{y}, \eta)$, it holds that

$$\int_{B\left(y,\frac{\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w \le \int_{B\left(\bar{y},\frac{3\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w = \left(\frac{3}{2}\right)^n \eta^n |B(0,1)| ⨍_{B\left(\bar{y},\frac{3\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w = o(\eta^n) \tag{3.8}$$

as $\eta \to 0^+$, and for $j = 2, \dots, k$,

$$\begin{aligned}\int_{B\left(y,\frac{(j-1)\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w &\le \int_{B\left(\bar{y},\frac{(j+1)\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w \\ &= \left(\frac{j+1}{2}\right)^n \eta^n |B(0,1)| ⨍_{B\left(\bar{y},\frac{(j+1)\eta}{2}\right)} \psi_{\mathcal{E}}(w)\,\mathrm{d}w = o(\eta^n)\end{aligned} \tag{3.9}$$

as $\eta \to 0^+$. From (3.7), (3.8), and (3.9), we conclude that if $\eta > 0$ is sufficiently small (depending on $\bar{y} \in \mathcal{F}$), then $B(\bar{y}, \eta) \subset \Omega$ and

$$\zeta(y) \ge \frac{1}{2}|B(0,1)|\eta^n$$

for all $y \in B(\bar{y}, \eta)$. Plugging this into (3.6) entails

$$\int_{B(\bar{y},\eta)} \omega\Big(\tfrac{k}{2}|f(y)|\Big)\,\mathrm{d}y \le \frac{2}{|B(0,1)|\eta^n} \int_\Omega \omega\Big(\tfrac{k}{2}|f(y)|\Big)\zeta(y)\,\mathrm{d}y < +\infty.$$

Finally, in view of the properties of ω there exists a linear lower bound on ω of the form $l : [0, +\infty) \to \mathbb{R}$, $l(t) = mt - c$ with $m, c > 0$. This implies

$$\int_{B(\bar{y},\eta)} |f(y)|\,\mathrm{d}y \le \frac{2}{km} \int_{B(\bar{y},\eta)} \omega\big(\tfrac{k}{2}|f(y)|\big) + c\,\mathrm{d}y < +\infty. \tag{3.10}$$

Hence, $f \in L^1(B(\bar{y}, \eta))$. □

Remark 3.2. (i) Let $p \in [1, +\infty)$ and assume that the function ω in Lemma 3.1 satisfies a p-coercivity condition, i.e.,

$$\omega(t) \ge mt^p - c \tag{3.11}$$

for $t \in [0, +\infty)$, with constants $m > 0$ and $c \ge 0$. Then we even obtain $f \in L^p_{\mathrm{loc}}(\Omega)$. To see this, simply replace (3.10) by an analogous reasoning with (3.11) instead of the lower bound l.

(ii) In the case of first-order differences, i.e., for $k = 1$, the proof of Lemma 3.1 can be shortened by choosing a simpler change of variables better suited for this context. Indeed, fix $x_0 \in \Omega$ and let $r_0 > 0$ be such that (3.1) holds with $k = 1$. Then, for almost all $x \in B(x_0, r_0)$,

$$\begin{aligned}+\infty > \int_{B(0,r_0)} \omega(|f(x+h) - f(x)|)\,\mathrm{d}h &= \int_{B(x,r_0)} \omega(|f(y) - f(x)|)\,\mathrm{d}y \\ &\ge \int_{B(x,r_0)} 2\,\omega(\tfrac{1}{2}|f(y)|)\,\mathrm{d}y - \omega(|f(x)|)|B(0,r_0)|,\end{aligned}$$

where we used (3.4). In analogy to (3.10) we derive $f \in L^1(B(x, r_0))$ for almost all $x \in B(x_0, r_0)$, and thus $f \in L^1(B(x_0, r_0))$.

To formulate a helpful implication of the previous lemma we make the following definition.

Definition 3.3. *We say that a family of mollifiers $(\rho_\varepsilon)_\varepsilon$ satisfies hypothesis (H), if for each $\varepsilon > 0$ there exist constants $\delta_\varepsilon > 0$ and $c_\varepsilon > 0$ such that $\rho_\varepsilon(r) \ge c_\varepsilon$ for all $r \in (0, \delta_\varepsilon)$.*

As the next example shows, a family of mollifiers satisfying (1.2) and (1.3) does not necessarily satisfy (H). Nevertheless, there is a number of interesting families of mollifiers that, in addition to (1.2) and (1.3), satisfy (H).

Example 3.4. For $0 < \varepsilon < 1$, let ρ^1_ε, ρ^2_ε, $\rho^3_\varepsilon : (0, +\infty) \to [0, +\infty)$ be the functions introduced in [5, Remark 8] by setting

$$\rho^1_\varepsilon(r) := \frac{\varepsilon}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\, r^{n-\varepsilon}} \chi_{(0,1)}(r), \quad \rho^2_\varepsilon(r) := \frac{n}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\, \varepsilon^n} \chi_{(0,\varepsilon)}(r),$$
$$\rho^3_\varepsilon(r) := \frac{1}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1}) |\log \varepsilon|\, r^n} \chi_{(\varepsilon,1)}(r),$$

for $r \in (0, +\infty)$. Then $(\rho^1_\varepsilon)_\varepsilon$ and $(\rho^2_\varepsilon)_\varepsilon$ satisfy (1.2), (1.3), and (H), whereas $(\rho^3_\varepsilon)_\varepsilon$ satisfies (1.2) and (1.3), but not (H).

Next we prove a consequence of Lemma 3.1 that, as announced before, allows us to remove the hypotheses of local integrability and local p-integrability of f in Theorem 1.4 and in Theorem 1.6, respectively, provided that the family $(\rho_\varepsilon)_\varepsilon$ satisfies condition (H).

Corollary 3.5. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ an open set, ω a function as in* (1.11)*, and $(\rho_\varepsilon)_\varepsilon$ a family of mollifiers satisfying* (1.2), (1.3)*, and* (H).

(i) *Suppose that $f : \Omega \to \mathbb{R}$ is a measurable function such that for every $x_0 \in \Omega$ there exists $r_0 > 0$ for which $B(x_0, (k+1)r_0) \subset \Omega$ and*
$$\limsup_{\varepsilon \to 0^+} \int_{B(x_0,r_0)} \int_{B(0,r_0)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x < +\infty. \tag{3.12}$$
Then, $f \in L^1_{\mathrm{loc}}(\Omega)$. In addition, if ω also fulfills (3.11) *for some $p \in [1, +\infty)$, then $f \in L^p_{\mathrm{loc}}(\Omega)$.*

(ii) *If $f : \Omega \to \mathbb{R}$ is a measurable function satisfying* (1.12)*, then $f \in L^1_{\mathrm{loc}}(\Omega)$.*

(iii) *Let $f : \Omega \to \mathbb{R}$ be a measurable function satisfying* (1.14) *and let ω meet in addition the coercivity condition* (3.11) *for some $p \in [1, +\infty)$. Then $f \in L^p_{\mathrm{loc}}(\Omega)$.*

Remark 3.6. In the case $k = 1$ Theorem 1.4 holds for f measurable without requiring the family $(\rho_\varepsilon)_\varepsilon$ to satisfy hypothesis (H). In fact, as suggested in [7], it suffices to replace f with $\arctan f$: if f satisfies (1.12), then so does $\arctan f$, which is a locally integrable function. On the other hand, if $\arctan f$ is constant, then so is f.

Proof. To prove (i), let $x_0 \in \Omega$. By assumption, there exist $r_0 > 0$ and $\varepsilon > 0$ such that the double integral
$$\int_{B(x_0,r_0)} \int_{B(0,r_0)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x$$
is finite. By hypothesis (H), we deduce that for $\tilde{r}_0 := \min\{\delta_\varepsilon, r_0, 1\}$,
$$\int_{B(x_0,r_0)} \int_{B(0,r_0)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \geq c_\varepsilon \int_{B(x_0,\tilde{r}_0)} \int_{B(0,\tilde{r}_0)} \omega\big(|\Delta^k_h f(x)|\big)\, \mathrm{d}h\, \mathrm{d}x.$$
Thus,
$$\int_{B(0,\tilde{r}_0)} \omega\big(|\Delta^k_h f(x)|\big)\, \mathrm{d}h < +\infty$$
for almost every $x \in B(x_0, \tilde{r}_0)$. From Lemma 3.1 and the arbitrariness of $x_0 \in \Omega$, we conclude that $f \in L^1_{\mathrm{loc}}(\Omega)$. If, in addition, ω satisfies (3.11) for some $p \in [1, +\infty)$, then by Remark 3.2 (i) we obtain $f \in L^p_{\mathrm{loc}}(\Omega)$.

Statement (ii) is an immediate consequence of (i).

To show (iii), we fix $x_0 \in \Omega$ and define $r_0 := \mathrm{dist}(x_0, \partial\Omega)/(2k)$. Then, $B(x_0, (k+1)r_0) \subset \Omega$. Moreover, for all $x \in B(x_0, r_0)$,
$$r_x = \frac{\mathrm{dist}(x, \partial\Omega)}{k} \geq \frac{\mathrm{dist}(x_0, \partial\Omega)}{k} - \frac{|x - x_0|}{k} \geq 2r_0 - \frac{r_0}{k} \geq r_0,$$
and consequently, (iii) follows from (1.14) in conjunction with (i). $\square$

4. Characterization of polynomials

This section is devoted to the proof of Theorem 1.4 and Proposition 1.3. As we will show at the end of Section 5, in the case where ω satisfies, in addition, (3.11) with $c = 0$, Theorem 1.4 is essentially a corollary of Theorem 1.6 and Remark 5.2. Nevertheless, we think that even in this case it is interesting to give an elementary proof without using the connection to Sobolev or BV spaces.

Proof of Theorem 1.4. The proof is divided into two steps, where we prove the assertion first for f smooth and then for f locally integrable.

Step 1: Let us assume first that $f \in C^\infty(\Omega)$.

Inspired by [7], we start by showing that there is a positive sequence $(r_j)_j$ with $r_j \to 0^+$ such that

$$\liminf_{j\to+\infty} \int_{B(x_0,r_0)} \omega\left(\frac{|\Delta^k_{r_j\theta} f(x)|}{|r_j\theta|^k}\right)\,\mathrm{d}x = 0 \tag{4.1}$$

for almost all $\theta \in \mathcal{S}^{n-1}$. Defining

$$\psi(h) := \int_{B(x_0,r_0)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right)\,\mathrm{d}x, \qquad h \in B(0,r_0)\setminus\{0\},$$

we have by hypothesis that

$$\lim_{\varepsilon\to0^+} \int_{B(0,r_0)} \psi(h)\rho_\varepsilon(|h|)\,\mathrm{d}h = 0.$$

A representation of the above expression in polar coordinates yields

$$\lim_{\varepsilon\to0^+} \int_0^{r_0} \bar\psi(r)\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r = 0 \qquad \text{with } \bar\psi(r) := \int_{\mathcal{S}^{n-1}} \psi(r\theta)\,\mathrm{d}\mathcal{H}^{n-1}(\theta). \tag{4.2}$$

We claim that one can find a positive sequence $(r_j)_j$ with $r_j \to 0^+$ such that

$$\lim_{j\to+\infty} \bar\psi(r_j) = 0. \tag{4.3}$$

Indeed, assume that there exist $0 < \delta < r_0$ and $c > 0$ such that for all $r \in [0,\delta]$ one has $\bar\psi(r) \geq c$. Then, in view of (4.2), (1.2), and (1.3),

$$\begin{aligned} 0 &= \lim_{\varepsilon\to0^+} \int_0^\delta \bar\psi(r)\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r \geq c \lim_{\varepsilon\to0^+} \int_0^\delta \rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r \\ &= \frac{c}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})} \lim_{\varepsilon\to0^+} \int_{B(0,\delta)} \rho_\varepsilon(|h|)\,\mathrm{d}h = \frac{c}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})} > 0, \end{aligned}$$

which is a contradiction. Thus (4.3) holds, and by Fatou's lemma, $\liminf_{j\to+\infty} \psi(r_j\theta) = 0$ for almost every $\theta \in \mathcal{S}^{n-1}$. This finishes the proof of (4.1).

Now we exploit the smoothness of f and apply the identity (2.4). Setting $h = r_j\theta$ with r_j and θ as in (4.1) gives

$$0 = \liminf_{j\to+\infty} \int_{B(x_0,r_0)} \omega\left(\left|f^{(k)}(x)\theta^k + \frac{\tilde R^{(k)}(r_j\theta;x)}{|r_j\theta|^k}\right|\right)\,\mathrm{d}x \geq \int_{B(x_0,r_0)} \omega(|f^{(k)}(x)\theta^k|)\,\mathrm{d}x.$$

Here we have used Fatou's lemma, the continuity of ω, and (2.5). In view of the remaining properties of the function ω, and using the fact that $f \in C^\infty(\Omega)$, we infer that

$$f^{(k)}(x)\theta^k = 0 \tag{4.4}$$

for every $x \in B(x_0,r_0)$. Since (4.4) holds for almost all $\theta \in \mathcal{S}^{n-1}$, the Taylor expansion for f about x_0 shows that f is a polynomial of degree at most $k-1$ in $B(x_0,r_0)$. The assertion follows, as Ω is connected.

Step 2: Assume that $f \in L^1_{\mathrm{loc}}(\Omega)$ and let $(\eta_\delta)_\delta$ be a family of standard mollifiers, i.e., $\eta_\delta \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\eta_\delta \subset B(0,\delta)$ for $\delta > 0$, $\eta_\delta \geq 0$, and $\int_{\mathbb{R}^n} \eta_\delta\,\mathrm{d}x = 1$.

We start by proving that the condition (1.12) is robust regarding convolution with η_δ, i.e., the smooth functions $f_\delta := f * \eta_\delta$ also satisfy (1.12) with some $\tilde r_0 > 0$, provided $\delta > 0$ is sufficiently small.

Let $x_0 \in \Omega$ and let $r_0 > 0$ be given by the hypothesis. Fix $\delta_0 \in (0, r_0)$ such that $x_0 \in \Omega_{\delta_0} := \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \delta_0\}$. Let $r_1 > 0$ be such that $B(x_0, (k+1)r_1) \subset \Omega_{\delta_0}$, and define $\tilde{r}_0 = \min\{r_1, r_0 - \delta_0\}$.

We observe that if $\delta \in (0, \delta_0)$, the difference quotient $\Delta_h^k f_\delta(x)$ is well-defined for all $x \in B(x_0, \tilde{r}_0)$ and $h \in B(0, \tilde{r}_0)$, and moreover,

$$|\Delta_h^k f_\delta(x)| \leq \int_{B(0,\delta)} |\Delta_h^k f(x-y)| \eta_\delta(y)\, \mathrm{d}y.$$

Since $\int_{B(0,\delta)} \eta_\delta(y)\, \mathrm{d}y = 1$, the set function $\mu_\delta(E) := \int_E \eta_\delta(y)\, \mathrm{d}y$, defined for all Borel sets $E \subset B(0,\delta)$, defines a probability measure on $B(0,\delta)$. Using the monotonicity and the convexity of ω, together with Jensen's inequality for the previous probability measure, leads to

$$\begin{aligned} \omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) &\leq \omega\left(\int_{B(0,\delta)} \frac{|\Delta_h^k f(x-y)|}{|h|^k} \eta_\delta(y)\, \mathrm{d}y\right) \\ &\leq \int_{B(0,\delta)} \omega\left(\frac{|\Delta_h^k f(x-y)|}{|h|^k}\right) \eta_\delta(y)\, \mathrm{d}y \end{aligned} \tag{4.5}$$

for $x \in B(x_0, \tilde{r}_0)$ and $h \in B(0, \tilde{r}_0) \setminus \{0\}$. Consequently, by Fubini's theorem,

$$\begin{aligned} &\int_{B(x_0,\tilde{r}_0)} \int_{B(0,\tilde{r}_0)} \omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &\quad\leq \int_{B(0,\delta)} \left(\int_{B(x_0,\tilde{r}_0)} \int_{B(0,\tilde{r}_0)} \omega\left(\frac{|\Delta_h^k f(x-y)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x\right) \eta_\delta(y)\, \mathrm{d}y. \end{aligned}$$

Performing the change of variables $z = x - y$ for fixed $y \in B(0,\delta)$, the previous triple integral is bounded by

$$\begin{aligned} &\int_{B(0,\delta)} \left(\int_{B(x_0,\tilde{r}_0+\delta)} \int_{B(0,\tilde{r}_0)} \omega\left(\frac{|\Delta_h^k f(z)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}z\right) \eta_\delta(y)\, \mathrm{d}y \\ &\quad\leq \int_{B(x_0,r_0)} \int_{B(0,r_0)} \omega\left(\frac{|\Delta_h^k f(z)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}z. \end{aligned}$$

We let $\varepsilon \to 0^+$ and account for the hypothesis on f to conclude that for $x_0 \in \Omega$ there exists $\delta_0 > 0$ such that f_δ satisfies (1.12) with $\tilde{r}_0 > 0$ for all $\delta \in (0, \delta_0)$.

Consequently, by Step 1 each f_δ with $\delta \in (0, \delta_0)$ is a polynomial of degree at most $k-1$ in $B(x_0, \tilde{r}_0)$. Since $f_\delta \to f$ pointwise almost everywhere in Ω as $\delta \to 0^+$, f coincides with a polynomial of degree at most $k-1$ almost everywhere in $B(x_0, \tilde{r}_0)$, and by a covering argument even almost everywhere in Ω. □

As already mentioned in the introduction, Proposition 1.3 follows from Theorem 1.4. The key is simply to specify the mollifiers $(\rho_\varepsilon)_\varepsilon$ in a suitable way, which was suggested in [5].

Proof of Proposition 1.3. Let $(\rho_\varepsilon^1)_\varepsilon$ be the family of mollifiers introduced in Example 3.4, that is,

$$\rho_\varepsilon^1(r) = \frac{\varepsilon}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\, r^{n-\varepsilon}} \chi_{(0,1)}(r), \quad r \in (0, +\infty).$$

We recall that $(\rho_\varepsilon^1)_\varepsilon$ satisfies (1.2), (1.3), and (H). Moreover, for each $x_0 \in \mathbb{R}^n$, we have that

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_{B(x_0,1)} \int_{B(0,1)} \frac{|\Delta_h^k f(x)|^p}{|h|^{kp}} \rho_\varepsilon^1(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &\quad\leq \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\mathcal{H}^{n-1}(\mathcal{S}^{n-1})} \left(\int_{B(x_0,1)} \int_{B(0,1)} \frac{|\Delta_h^k f(x)|^p}{|h|^{n+kp}}\, \mathrm{d}h\, \mathrm{d}x\right) = 0, \end{aligned} \tag{4.6}$$

owing to the boundedness of the double integral in brackets by (1.10).

Applying Corollary 3.5 (i) with $\omega(t) = t^p$ for $t \in [0, +\infty)$, we conclude that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, which together with (4.6), allows us to apply Theorem 1.4 with the same function ω. Consequently, f coincides with a polynomial of degree at most $k-1$ almost everywhere in $\mathbb{R}^n$. □

5. Characterization of Sobolev spaces

In this section we prove Theorem 1.6 and, as a corollary, we also provide an alternative proof of Theorem 1.4 in the case where ω satisfies, in addition, condition (3.11) with $c = 0$. The basic idea of the proof of Theorem 1.6 is to approximate f by smooth functions, for which we establish the desired relations using the classical Taylor formula. Several of our arguments were inspired by those in [5, 11].

We start by mentioning that the constant $\overline{K}_{n,p,k}$ in (1.15) is given through the minimization problem

$$\overline{K}_{n,p,k} := \min\left\{ \fint_{\mathcal{S}^{n-1}} |\sigma_k(h)F\,h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h) \colon F \in \mathbb{M}_{n^{(k-1)}n}, |F| = 1 \right\}, \tag{5.1}$$

with the notations as in Section 2. Note that $\overline{K}_{n,p,k} \in (0,1]$.

Remark 5.1. If $k = 1$ and $F \in \mathbb{M}_{1n}$, then

$$\fint_{\mathcal{S}^{n-1}} |\sigma_1(h)F\,h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h) = \fint_{\mathcal{S}^{n-1}} |F^T \cdot h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h).$$

On the other hand, $\fint_{\mathcal{S}^{n-1}} |e \cdot h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h) = \fint_{\mathcal{S}^{n-1}} |e' \cdot h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h)$ for all $e, e' \in \mathcal{S}^{n-1}$. Hence, $\overline{K}_{n,p,1} = K_{n,p} = \fint_{\mathcal{S}^{n-1}} |e \cdot h|^p \,\mathrm{d}\mathcal{H}^{n-1}(h)$, where e is any unit vector in $\mathbb{R}^n$ and $K_{n,p}$ is the constant in Theorem 1.2. For $k > 1$, the invariance of the integral in the definition of $\overline{K}_{n,p,k}$ is, in general, no longer true. For instance, take $k = 2$, $n = 2$, and consider the two matrices $F = \frac{1}{\sqrt{2}}\mathrm{diag}\,(1,1)$ and $F' = \frac{1}{\sqrt{2}}\mathrm{diag}\,(1,-1)$. Then, $\fint_{\mathcal{S}^1} |\sigma_2(h)F\,h|^p \,\mathrm{d}\mathcal{H}^1(h) = 2^{-p/2}\fint_{\mathcal{S}^1} |h|^{2p} \,\mathrm{d}\mathcal{H}^1(h) = 2^{-p/2} > 2^{-p/2}\fint_{\mathcal{S}^1} |h_1^2 - h_2^2|^p \,\mathrm{d}\mathcal{H}^1(h) = \fint_{\mathcal{S}^1} |\sigma_2(h)F'\,h|^p \,\mathrm{d}\mathcal{H}^1(h)$.

Proof of Theorem 1.6. The proofs for the necessary and sufficient condition are presented in two steps. Then, (1.15) and the estimates (1.16) will follow directly from (5.2) and (5.17).

Step 1: We prove that if (1.14) holds, then $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$ with $D^k f \in L^p(\Omega;\mathbb{R}^{n^k})$ and

$$\begin{aligned} m\overline{K}_{n,p,k} \int_\Omega |D^k f(x)|^p \,\mathrm{d}x &\le \int_\Omega \fint_{\mathcal{S}^{n-1}} \omega\left(|\sigma_k(h)\nabla(D^{k-1}f)(x)\,h|\right) \,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0^+} \int_\Omega \int_{B(0,r_x)} \omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x. \end{aligned} \tag{5.2}$$

Substep 1.1: Let $(\eta_\delta)_\delta$ be a family of standard smooth mollifiers as in Step 2 of the proof of Theorem 1.4 (see Section 4). For $\delta > 0$ let $f_\delta := f * \eta_\delta$, defined on $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \delta\}$. We prove that each f_δ satisfies the inequality

$$\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})} \omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x \le \int_\Omega\int_{B(0,r_x)} \omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x,$$

where $r_{x,\delta} := \mathrm{dist}(x,\partial\Omega_\delta)/k$ and $\varepsilon > 0$ is fixed.

As in (4.5), we have that for all $x \in \Omega_\delta$ and for all $h \in B(0,r_{x,\delta})\backslash\{0\}$,

$$\omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) \le \int_{B(0,\delta)} \omega\left(\frac{|\Delta_h^k f(x-y)|}{|h|^k}\right) \eta_\delta(y)\,\mathrm{d}y.$$

Consequently, the change of variables $z := x - y$ for $y \in B(0,\delta)$ implies

$$\begin{aligned} &\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})} \omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x \\ &\le \int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\int_{B(0,\delta)} \omega\left(\frac{|\Delta_h^k f(x-y)|}{|h|^k}\right) \eta_\delta(y)\rho_\varepsilon(|h|)\,\mathrm{d}y\,\mathrm{d}h\,\mathrm{d}x \\ &\le \int_{\Omega}\int_{B(0,r_z)}\int_{B(0,\delta)} \omega\left(\frac{|\Delta_h^k f(z)|}{|h|^k}\right) \eta_\delta(y)\rho_\varepsilon(|h|)\,\mathrm{d}y\,\mathrm{d}h\,\mathrm{d}z \\ &= \int_{\Omega}\int_{B(0,r_z)} \omega\left(\frac{|\Delta_h^k f(z)|}{|h|^k}\right) \rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}z, \end{aligned}$$

where in the second inequality we also used the simple geometrical inclusion

$$\begin{aligned}&\{(x,h,y)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n:\ x\in\Omega_\delta,\ h\in B(0,r_{x,\delta}),\ y\in B(0,\delta)\}\\&\quad\subset\{(z+y,h,y)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n:\ z\in\Omega,\ h\in B(0,r_z),\ y\in B(0,\delta)\}.\end{aligned}$$

This concludes Substep 1.1.

Substep 1.2: Let f_δ be as in Substep 1.1. We prove that

$$\begin{aligned}&\int_{\Omega_\delta}\fint_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x\\&\qquad\le\liminf_{\varepsilon\to0^+}\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x.\end{aligned}\tag{5.3}$$

Fix $\delta>0$ and let $\Omega'_\delta\subset\subset\Omega_\delta$. In view of (2.8) applied locally, one finds for all $x\in\Omega'_\delta$ and $h\in B(0,r'_{x,\delta})$, with $r'_{x,\delta}:=\operatorname{dist}(x,\partial\Omega'_\delta)/k$, that

$$\Delta_h^k f_\delta(x)=\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h+\tilde R_\delta^{(k)}(h;x),\tag{5.4}$$

and by (2.3) there exists a positive constant c_δ depending only on k and $\|f_\delta\|_{C^{k+1}(\overline{\Omega'_\delta})}$ such that

$$\left|\tilde R_\delta^{(k)}(x;h)\right|\le c_\delta|h|^{k+1}.\tag{5.5}$$

It follows from Schwartz's inequality and (2.6) that for all $x\in\Omega'_\delta$ and $h\in B(0,r'_{x,\delta})\backslash\{0\}$,

$$\frac{|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|}{|h|^k}\le|D^kf_\delta(x)|\le\|f_\delta\|_{C^{k+1}(\overline{\Omega'_\delta})},$$

and, by (5.4) and (5.5), also

$$\frac{|\Delta_h^kf_\delta(x)|}{|h|^k}\le\|f_\delta\|_{C^{k+1}(\overline{\Omega'_\delta})}+c_\delta\operatorname{diam}\Omega'_\delta=:C_\delta.$$

Thus, due to the local Lipschitz continuity of ω, there exists a constant L_δ, depending only on C_δ, such that

$$\left|\omega\left(\frac{|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|}{|h|^k}\right)-\omega\left(\frac{|\Delta_h^kf_\delta(x)|}{|h|^k}\right)\right|\le L_\delta c_\delta|h|$$

for all $x\in\Omega'_\delta$ and $h\in B(0,r'_{x,\delta})\backslash\{0\}$, where once again we used (5.4) and (5.5).

Consequently, one obtains that

$$\begin{aligned}&\int_{\Omega'_\delta}\int_{B(0,r'_{x,\delta})}\omega\left(\frac{|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x\\&\quad\le\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\omega\left(\frac{|\Delta_h^kf_\delta(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x+L_\delta c_\delta|\Omega'_\delta|\int_{B(0,\operatorname{diam}\Omega'_\delta)}|h|\rho_\varepsilon(|h|)\,\mathrm{d}h\end{aligned}\tag{5.6}$$

for every $\varepsilon>0$. We observe that the second term in (5.6) vanishes in the limit $\varepsilon\to0^+$, since

$$\lim_{\varepsilon\to0^+}\int_{B(0,\operatorname{diam}\Omega'_\delta)}|h|\rho_\varepsilon(|h|)\,\mathrm{d}h=0.\tag{5.7}$$

Indeed, for all $0<\gamma<\operatorname{diam}\Omega'_\delta$, we have that

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\int_{B(0,\operatorname{diam}\Omega'_\delta)}|h|\rho_\varepsilon(|h|)\,\mathrm{d}h\\&\qquad\le\limsup_{\varepsilon\to0^+}\bigg(\gamma\int_{B(0,\gamma)}\rho_\varepsilon(|h|)\,\mathrm{d}h+\int_{B(0,\operatorname{diam}\Omega'_\delta)\backslash B(0,\gamma)}|h|\rho_\varepsilon(|h|)\,\mathrm{d}h\bigg)\\&\qquad\le\gamma+(\operatorname{diam}\Omega'_\delta)\lim_{\varepsilon\to0^+}\int_{\{|h|>\gamma\}}\rho_\varepsilon(|h|)\,\mathrm{d}h=\gamma,\end{aligned}$$

where we used (1.2) and (1.3). Letting $\gamma\to0^+$, we obtain (5.7).

Next we prove that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{\Omega'_\delta}\int_{B(0,r'_{x,\delta})}&\omega\left(\frac{|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x\\&=\int_{\Omega'_\delta}\fint_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x.\end{aligned}\tag{5.8}$$

Let $x\in\Omega'_\delta$. Writing $h\in B(0,r'_{x,\delta})$ in spherical coordinates, i.e., $h=r\theta$ with $\theta\in\mathcal{S}^{n-1}$ and $r>0$, and using the $(k-1)$-homogeneity of σ_k results in

$$\begin{aligned}&\int_{B(0,r'_{x,\delta})}\omega\left(\frac{|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\\&=\int_0^{r'_{x,\delta}}\int_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(\theta)\nabla(D^{k-1}f_\delta)(x)\,\theta|\right)\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}(\theta)\,\mathrm{d}r\\&=\left(\int_0^{r'_{x,\delta}}\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r\right)\int_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(\theta)\nabla(D^{k-1}f_\delta)(x)\,\theta|\right)\,\mathrm{d}\mathcal{H}^{n-1}(\theta)\\&=\left(1-\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\int_{r'_{x,\delta}}^{+\infty}\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r\right)\fint_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(\theta)\nabla(D^{k-1}f_\delta)(x)\,\theta|\right)\,\mathrm{d}\mathcal{H}^{n-1}(\theta).\end{aligned}\tag{5.9}$$

Here we used the equalities

$$1=\int_{\mathbb{R}^n}\rho_\varepsilon(|h|)\,\mathrm{d}h=\mathcal{H}^{n-1}(\mathcal{S}^{n-1})\int_0^{+\infty}\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r.\tag{5.10}$$

Moreover, we observe that

$$\lim_{\varepsilon\to 0^+}\int_{r'_{x,\delta}}^{+\infty}\rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r=0$$

by (1.3). Then, equality (5.8) is achieved by integrating (5.9) over Ω'_δ and using Lebesgue's dominated convergence theorem, taking into account that ω is locally bounded.

In view of (5.6), (5.7), and (5.8), we finally obtain

$$\begin{aligned}\int_{\Omega'_\delta}\fint_{\mathcal{S}^{n-1}}&\omega\left(|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x\\&\le\liminf_{\varepsilon\to 0^+}\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x,\end{aligned}$$

from which (5.3) follows by letting $\Omega'_\delta\nearrow\Omega_\delta$ under consideration of Lebesgue's monotone convergence theorem.

Substep 1.3: We conclude the proof of Step 1.

Let f_δ and Ω_δ be as in Substep 1.1. Let $\Omega'\subset\subset\Omega$ and consider $\delta>0$ sufficiently small such that $\Omega'\subset\Omega_\delta$. Then, by Substep 1.2,

$$\begin{aligned}\int_{\Omega'}\fint_{\mathcal{S}^{n-1}}&\omega\left(|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x\\&\le\liminf_{\varepsilon\to 0^+}\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x.\end{aligned}\tag{5.11}$$

On the one hand, by the definition of $\overline{K}_{n,p,k}$ and (1.13),

$$\begin{aligned}m\,\overline{K}_{n,p,k}\int_{\Omega'}|D^k f_\delta(x)|^p\,\mathrm{d}x&\le m\int_{\Omega'}\fint_{\mathcal{S}^{n-1}}|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|^p\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x\\&\le\int_{\Omega'}\fint_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)\nabla(D^{k-1}f_\delta)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x.\end{aligned}\tag{5.12}$$

On the other hand, by Substep 1.1 and the hypothesis, it follows that

$$\begin{aligned}\liminf_{\varepsilon\to0^+}\int_{\Omega_\delta}\int_{B(0,r_{x,\delta})}\omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x\\ \leq\liminf_{\varepsilon\to0^+}\int_{\Omega}\int_{B(0,r_x)}\omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x<+\infty.\end{aligned}\tag{5.13}$$

Therefore, from (5.11), (5.12), and (5.13), we infer that

$$m\,\overline{K}_{n,p,k}\int_{\Omega'}|D^k f_\delta(x)|^p\,\mathrm{d}x\leq\liminf_{\varepsilon\to0^+}\int_{\Omega}\int_{B(0,r_x)}\omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x<+\infty.\tag{5.14}$$

In particular, $(D^k f_\delta)_\delta$ is uniformly bounded in $L^p(\Omega';\mathbb{R}^{n^k})$ and thus, up to a not relabeled subsequence, $D^k f_\delta \rightharpoonup g$ weakly in $L^p(\Omega';\mathbb{R}^{n^k})$ for some $g\in L^p(\Omega';\mathbb{R}^{n^k})$. Since $f_\delta\to f$ in $L^p_{\mathrm{loc}}(\Omega)$, the kth-order distributional derivative of f, i.e., $D^k f$, coincides with g. Moreover, using once more the definition of $\overline{K}_{n,p,k}$ and (1.13), by the sequential weak lower semicontinuity in L^p of the nonnegative convex functional

$$W\mapsto\int_{\Omega'}⨍_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)W\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x,\qquad W\in L^p(\Omega';\mathbb{M}_{n^{(k-1)}n}),$$

together with (5.11) and (5.13), we conclude that

$$\begin{aligned}m\,\overline{K}_{n,p,k}\int_{\Omega'}|D^k f(x)|^p\,\mathrm{d}x&\leq\int_{\Omega'}⨍_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)\nabla(D^{k-1}f)(x)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}x\\ &\leq\liminf_{\varepsilon\to0^+}\int_{\Omega}\int_{B(0,r_x)}\omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x<+\infty.\end{aligned}$$

Letting $\Omega'\nearrow\Omega$, in view of Lebesgue's monotone convergence theorem we obtain (5.2). In particular, $D^k f\in L^p(\Omega;\mathbb{R}^{n^k})$. To finish the proof of Step 1, it remains to show that $f\in W^{k,p}_{\mathrm{loc}}(\Omega)$ for $k>1$ ($k=1$ follows from the hypothesis $f\in L^p_{\mathrm{loc}}(\Omega)$ and (5.2)).

By Gagliardo-Nirenberg's interpolation inequalities (see, for instance, [16, Lemma 4.2.2]), given $q\geq1$, $l\in\{1,\cdots,k-1\}$, and an arbitrary ball B compactly contained in Ω, there exists a constant C, only depending on n,q,l,k, and B, such that

$$\int_B|D^l f_\delta(x)|^q\,\mathrm{d}x\leq C\left(\int_B|f_\delta(x)|^q\,\mathrm{d}x+\int_B|D^k f_\delta(x)|^q\,\mathrm{d}x\right).\tag{5.15}$$

Using (5.15) with $q=p$, (5.14), and the hypothesis $f\in L^p_{\mathrm{loc}}(\Omega)$ together with the properties of the smooth mollifiers implies that for all $l\in\{1,\cdots,k-1\}$,

$$\sup_{\delta>0}\int_B|D^l f_\delta(x)|^p\,\mathrm{d}x\leq\bar{C},\tag{5.16}$$

where $\bar{C}$ is a positive constant only depending on $n,p,l,k,B,\|f\|_{L^p(B)}$, and on the finite limit in (1.14). Consequently, since $B\subset\subset\Omega$ was taken arbitrarily, we conclude that $f\in W^{k,p}_{\mathrm{loc}}(\Omega)$.

Step 2: We prove that if $f\in W^{k,p}_{\mathrm{loc}}(\Omega)$ with $D^k f\in L^p(\Omega;\mathbb{R}^{n^k})$, then (1.14) holds and

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\int_{\Omega}\int_{B(0,r_x)}\omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x\\ &\quad\leq\int_{\Omega}⨍_{\mathcal{S}^{n-1}}\omega\left(|\sigma_k(h)\nabla(D^{k-1}f)(y)\,h|\right)\,\mathrm{d}\mathcal{H}^{n-1}(h)\,\mathrm{d}y\leq M\int_{\Omega}|D^k f(y)|^p\,\mathrm{d}y.\end{aligned}\tag{5.17}$$

Let $\varepsilon>0$ be fixed and for $\delta>0$ let $f_\delta=f*\eta_\delta\in C^\infty(\Omega_\delta)$ be as in Substep 1.1. Then, in particular,

$$D^k f_\delta\to D^k f\qquad\text{in } L^p_{loc}(\Omega),\tag{5.18}$$

and

$$f_\delta(x)\to f(x)\quad\text{and}\quad D^k f_\delta(x)\to D^k f(x),\tag{5.19}$$

for almost every $x\in\Omega$, as $\delta\to0^+$.

From the fundamental theorem of calculus we conclude that for each $x \in \Omega_\delta$ and $h \in B(0, r_{x,\delta})$,

$$\begin{aligned}\Delta_h^k f_\delta(x) &= \int_{(0,1)^k} \frac{\partial^k}{\partial s_1 \cdots \partial s_k} f_\delta(x + s_1 h + \cdots + s_k h)\, \mathrm{d}s_1 \cdots\, \mathrm{d}s_k \\ &= \int_{(0,1)^k} \sigma_k(h) \nabla(D^{k-1} f_\delta)(x + s_1 h + \cdots + s_k h)\, h\, \mathrm{d}s_1 \cdots\, \mathrm{d}s_k.\end{aligned} \tag{5.20}$$

The last equality follows from a straightforward calculation of the involved partial derivatives under consideration of the notation introduced in Section 2.

Based on the identity (5.20) we now use the monotonicity and convexity of ω together with Jensen's inequality, Tonellis's theorem, the change of variables $y = x + s_1 h + \cdots + s_k h$ for $h \in B(0, r_{x,\delta})$ and $s_i \in (0,1)$, $i \in \{1, \cdots, k\}$, with Jacobian determinant 1, and the inclusion

$$\begin{aligned}&\left\{(s_1, \cdots, s_k, x, h) \in (0,1)^k \times \mathbb{R}^n \times \mathbb{R}^n :\ x \in \Omega_\delta,\ h \in B(0, r_{x,\delta})\right\} \\ &\quad \subset \left\{(s_1, \cdots, s_k, y - s_1 h - \cdots - s_k h, h) \in (0,1)^k \times \mathbb{R}^n \times \mathbb{R}^n :\ y \in \Omega_\delta,\ h \in \mathbb{R}^n\right\},\end{aligned}$$

to derive

$$\begin{aligned}&\int_{\Omega_\delta} \int_{B(0, r_{x,\delta})} \omega\left(\frac{|\Delta_h^k f_\delta(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &\le \int_{\Omega_\delta} \int_{B(0, r_{x,\delta})} \int_{(0,1)^k} \left[\omega\left(\frac{|\sigma_k(h) \nabla(D^{k-1} f_\delta)(x + s_1 h + \cdots + s_k h)\, h|}{|h|^k}\right)\right. \\ &\qquad\qquad \left.\rho_\varepsilon(|h|)\right] \mathrm{d}s_1 \cdots\, \mathrm{d}s_k\, \mathrm{d}h\, \mathrm{d}x \\ &\le \int_{\Omega_\delta} \int_{\mathbb{R}^n} \omega\left(\frac{|\sigma_k(h) \nabla(D^{k-1} f_\delta)(y)\, h|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}y \\ &= \int_{\Omega_\delta} \fint_{\mathcal{S}^{n-1}} \omega\left(|\sigma_k(h) \nabla(D^{k-1} f_\delta)(y)\, h|\right)\, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}y \le \int_{\Omega_\delta} \omega\left(|D^k f_\delta(y)|\right)\, \mathrm{d}y \\ &\le M \int_{\Omega_\delta} |D^k f_\delta(y)|^p\, \mathrm{d}y \le M \int_\Omega |D^k f(y)|^p\, \mathrm{d}y.\end{aligned}$$

Notice that also (5.10), the $(k-1)$-homogeneity of σ_k together with Schwartz's inequality, the growth condition (1.13), and the properties of the smooth mollifiers were exploited in the foregoing estimate.

Passing these inequalities to the limit as $\delta \to 0^+$ and using Fatou's lemma and the Vitali-Lebesgue convergence theorem together with the continuity of ω, (5.18), and (5.19) we get for a fixed $\Omega' \subset\subset \Omega$ and $r'_x := \mathrm{dist}(x, \partial\Omega')/k$ that

$$\begin{aligned}&\int_{\Omega'} \int_{B(0, r'_x)} \omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x \\ &\quad \le \int_{\Omega'} \fint_{\mathcal{S}^{n-1}} \omega\left(|\sigma_k(h) \nabla(D^{k-1} f)(y)\, h|\right)\, \mathrm{d}\mathcal{H}^{n-1}(h)\, \mathrm{d}y \le M \int_\Omega |D^k f(y)|^p\, \mathrm{d}y.\end{aligned}$$

Finally, Step 2 is achieved by letting $\Omega' \nearrow \Omega$ and $\varepsilon \to 0^+$. □

Remark 5.2. *The case* $p = 1$. Assume that the hypotheses of Theorem 1.6 are fulfilled for $p = 1$. Arguing as in Step 1 of the proof of Theorem 1.6, we conclude that if (1.14) holds, then $f \in W^{k-1,1}_{\mathrm{loc}}(\Omega)$ and $D^k f \in \mathcal{M}(\Omega; \mathbb{R}^{n^k})$, i.e., $D^k f$ is a Radon measure with finite total variation in Ω. Moreover, denoting by $|D^k f|(\Omega)$ the total variation of $D^k f$ in Ω,

$$m \overline{K}_{n,1,k} |D^k f|(\Omega) \le \liminf_{\varepsilon \to 0^+} \int_\Omega \int_{B(0, r_x)} \omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right) \rho_\varepsilon(|h|)\, \mathrm{d}h\, \mathrm{d}x.$$

Indeed, from (5.14) and (5.15) with $q = 1$ we infer (5.16) for $p = 1$. Using, in addition, the compact embedding of $BV(B; \mathbb{R}^m)$ into $L^1(B; \mathbb{R}^m)$, and the sequential lower semicontinuity of the total variation with respect to weak-$\star$ convergence in $\mathcal{M}(\Omega; \mathbb{R}^m)$, the statement follows.

We also observe that the arguments in Step 2 of the previous proof imply that if $f \in W^{k,1}_{\rm loc}(\Omega)$ is such that $D^k f \in L^1(\Omega;\mathbb{R}^{n^k})$, then (1.14) and (5.17) hold with $p = 1$.

Finally, we show that Theorem 1.4 can be considered as a corollary of Theorem 1.6 under an additional condition on the function ω.

Alternative proof of Theorem 1.4 under condition (3.11) *with* $c = 0$. Fix $x_0 \in \Omega$, and let $r_0 > 0$ be such that $B(x_0,(k+1)r_0) \subset \Omega$ and (1.12) holds. With the definition $r_x := \operatorname{dist}(x, \partial B(x_0, \frac{r_0}{2}))/k$ one obtains

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0^+} \int_{B(x_0,r_0)}\int_{B(0,r_0)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x\\
&\geq \limsup_{\varepsilon\to 0^+} \int_{B\left(x_0,\frac{r_0}{2}\right)}\int_{B(0,r_x)} \omega\left(\frac{|\Delta^k_h f(x)|}{|h|^k}\right)\rho_\varepsilon(|h|)\,\mathrm{d}h\,\mathrm{d}x.
\end{aligned}
$$

So, in view of Step 1 of the previous proof, see (5.2) for $p \in (1,+\infty)$ and Remark 5.2 for $p = 1$, where only the lower bound in (1.13) was used, we conclude that $f \in W^{k,p}_{\rm loc}(B(x_0,\frac{r_0}{2}))$ with $D^k f \in L^p(B(x_0,\frac{r_0}{2});\mathbb{R}^{n^k})$ and

$$
\int_{B\left(x_0,\frac{r_0}{2}\right)} |D^k f(x)|^p\,\mathrm{d}x = 0, \tag{5.21}
$$

if $p \in (1,+\infty)$, and $f \in W^{k-1,1}_{\rm loc}(B(x_0,\frac{r_0}{2}))$, $D^k f \in \mathcal{M}(B(x_0,\frac{r_0}{2});\mathbb{R}^{n^k})$, and

$$
|D^k f|\left(B\left(x_0, r_0/2\right)\right) = 0, \tag{5.22}
$$

if $p = 1$. Equalities (5.21) and (5.22), together with a covering argument, yield that f coincides almost everywhere in Ω with a polynomial of degree smaller than or equal to $k - 1$. $\square$

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References

[1] Aubert, G., and Kornprobst, P. Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? *SIAM J. Numer. Anal. 47*, 2 (2009), 844–860.

[2] Borghol, R. Some properties of Sobolev spaces. *Asymptotic Anal. 51*, 3-4 (2007), 303–318.

[3] Bourgain, J., Brézis, H., and Mironescu, P. Another look at Sobolev spaces. In *Optimal control and partial differential equations. In honour of Professor Alain Bensoussan's 60th birthday. Proceedings of the conference, Paris, France, December 4, 2000.* Amsterdam: IOS Press; Tokyo: Ohmsha, 2001, pp. 439–455.

[4] Bourgain, J., Brézis, H., and Mironescu, P. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math. 87* (2002), 77–101.

[5] Brézis, H. How to recognize constant functions. Connections with Sobolev spaces. *Russ. Math. Surv. 57*, 4 (2002), 693–708.

[6] Dávila, J. On an open question about functions of bounded variation. *Calc. Var. Partial Differ. Equ. 15*, 4 (2002), 519–527.

[7] De Marco, G., Mariconda, C., and Solimini, S. An elementary proof of a characterization of constant functions. *Adv. Nonlinear Stud. 8*, 3 (2008), 597–602.

[8] Gilboa, G., and Osher, S. Nonlocal linear image regularization and supervised segmentation. *Multiscale Model. Simul. 6*, 2 (2007), 595–630.

[9] Gilboa, G., and Osher, S. Nonlocal operators with applications to image processing. *Multiscale Model. Simul. 7*, 3 (2008), 1005–1028.
[10] Ignat, R. On an open problem about how to recognize constant functions. *Houston J. Math. 31*, 1 (2005), 285–304.
[11] Leoni, G., and Spector, D. Characterization of Sobolev and BV spaces. *J. Funct. Anal. 261*, 10 (2011), 2926–2958.
[12] Leoni, G., and Spector, D. Corrigendum to "Characterization of Sobolev and *BV* spaces" [J. Funct. Anal. 261 (10) (2011) 2926–2958]. *J. Funct. Anal. 266*, 2 (2014), 1106–1114.
[13] Nguyen, H.-M. Γ-convergence and Sobolev norms. *C. R., Math., Acad. Sci. Paris 345*, 12 (2007), 679–684.
[14] Nguyen, H.-M. Further characterizations of Sobolev spaces. *J. Eur. Math. Soc. (JEMS) 10*, 1 (2008), 191–229.
[15] Stein, E. M., and Zygmund, A. On the differentiability of functions. *Stud. Math. 23* (1964), 247–283.
[16] Ziemer, W. P. *Weakly differentiable functions*, vol. 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

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